Basic Proofs

September 22, 2009

1 Direct Proofs

Prove that the product of two consecutive integers is even.

Proof Let a and $a+1$ be two consecutive integers. Since they are consecutive, one must be even and one must be odd. Let a be even and $a+1$ be odd. By definitions of odd and even, $a = 2k$ and $a+1 = 2k+1$ for some $k \in \mathbb{Z}$. Thus, $a(a+1) = 2k(2k+1) = 4k^2 + 2k = 2(2k^2 + k)$. Now, by letting $z = 2k^2 + k$, we have that $a(a+1) = 2z$ for some $z \in \mathbb{Z}$. Therefore, product of two consecutive integers is indeed even. ■

Prove that $n^3 + 3n^2 + 2n$ is even for all integers n.

Proof By inspection, it is possible to factor $n^3 + 3n^2 + 2n$ into $n(n^2 + 3n + 2)$. By factoring further, we obtain $n(n+1)(n+2)$. We have a product of three consecutive numbers, so at least one of the numbers must be even. Since we have already proven the product of two consecutive integers is even, then we can say that $n(n+1)(n+2) = n(2k)$ for some$k \in \mathbb{Z}$. We could rewrite this as $2nk$, and by letting $z = nk$, we have that $n^3 + 3n^2 + 2n = 2z$ for some $z \in \mathbb{Z}$. ■

Prove that for all non-negative real numbers x and y, $\frac{x+y}{2} \geq \sqrt{xy}$.

Proof

Since $x \geq 0$ and $y \geq 0$, then $x - y$ is a real number such that $(x-y)^2 \geq 0$. By expansion, we obtain $x^2 - 2xy + y^2 \geq 0$. By adding $4xy$ to both sides, we obtain the inequality $x^2 + 2xy + y^2 \geq 4xy$. By factoring, we obtain $(x+y)^2 \geq 4xy$. By squaring both sides, we obtain $x + y \geq 2\sqrt{xy}$. By dividing both sides by 2, we obtain $\frac{x+y}{2} \geq \sqrt{xy}$. ■

Let x and y be 2 rational numbers. Prove that xy is a rational number.

Proof Since x and y are rational numbers, we can say that $x = \frac{m}{n}$ and $y = \frac{p}{q}$, where $m, n, p, q \in \mathbb{Z}$, $n \neq 0$, and $q \neq 0$. By substitution, we have that $xy = \frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq}$. By letting $a = mp$ and $b = nq$, we obtain $xy = \frac{a}{b}$ for two integers a and b where $b \neq 0$. Hence, xy is a rational number. ■

Let x and y be 2 rational numbers. Prove that $x + y$ is a rational number.

Proof Since x and y are rational numbers, we can say that $x = \frac{m}{n}$ and $y = \frac{p}{q}$, where $m, n, p, q \in \mathbb{Z}$, $n \neq 0$, and $q \neq 0$. By substitution, we have that $x + y = \frac{m}{n} + \frac{p}{q} = \frac{mq}{nq} + \frac{np}{nq} = \frac{mq + np}{nq}$. By letting $a = mp + np$ and $b = nq$, we obtain $x + y = \frac{a}{b}$ for two integers a and b where $b \neq 0$. Hence, $x + y$ is a rational number. ■

Let a and b be perfect squares. Prove that ab is a perfect square.

Proof Since a and b are perfect squares, we can say that $a = c^2$ and $b = d^2$ for some $c, d \in \mathbb{Z}$. By substitution, we have that $ab = c^2d^2 = (cd)^2$. By letting $j = cd$, we obtain $ab = j^2$ for some $j \in \mathbb{Z}$. Hence, ab is a perfect square. ■

Let a and b be perfect squares. Prove that $a + 2\sqrt{ab} + b$ is a perfect square.

Proof Since a and b are perfect squares, we can say that $a = c^2$ and $b = d^2$ for some $c, d \in \mathbb{Z}$. By substitution, we have that $a+2\sqrt{ab}+b = c^2+2\sqrt{c^2d^2}+d^2 = c^2+2\sqrt{(cd)^2}+d^2 = c^2+2cd+d^2 = (c+d)^2$. By letting $j = c+d$, we obtain $a+2\sqrt{ab}+b = j^2$ for some $j \in \mathbb{Z}$. Hence, $a+b+2\sqrt{ab}$ is a perfect square. ■

Let a and b be perfect cubes. Prove that ab is a perfect cube.

Proof Since a and b are perfect cubes, we can say that $a = c^3$ and $b = d^3$ for some $c, d \in \mathbb{Z}$. By substitution, we have that $ab = c^3d^3 = (cd)^3$. By letting $j = cd$, we obtain $ab = j^3$ for some $j \in \mathbb{Z}$. Hence, ab is a perfect cube. ■

Let a and b be perfect cubes. Prove that $a + \sqrt[3]{a^2b} + \sqrt[3]{ab^2} + b$ is a perfect cube.

Proof Since a and b are perfect cubes, we can say that $a = c^3$ and $b = d^3$ for some $c, d \in \mathbb{Z}$. By substitution, we have that $a + \sqrt[3]{a^2b} + \sqrt[3]{ab^2} + b = c^3 + \sqrt[3]{(c^3)^2d^3} + \sqrt[3]{c^3(d^3)^2} + d^3 = c^3 + \sqrt[3]{c^6d^3} + \sqrt[3]{c^3d^6} + d^3 = c^3 + c^2d + cd^2 + d^3 = (c+d)^3$. By letting $j = c+d$, we obtain $a + \sqrt[3]{a^2b} + \sqrt[3]{ab^2} + b = j^3$ for some $j \in \mathbb{Z}$. Hence, $a + \sqrt[3]{a^2b} + \sqrt[3]{ab^2} + b$ is a perfect cube. ■

Prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all positive integers n.

Proof Let $S = 1+2+3+...+(n-2)+(n-1)+n$. By commutativity of addition, we obtain $S = n+(n-1)+(n-2)+...+3+2+1$. By adding the two sums together, we obtain $2S = n(n+1)$, so $S = \frac{n(n+1)}{2}$. ■

2 Proof By Cases

Prove that $\forall x, y \in \mathbb{R}, |xy| = |x|\,|y|$.

Proof

Case 1: Let $x \geq 0$ and $y \geq 0$. Then $|xy| = xy$. Also, $|x| = x$ and $|y| = y$, so $|x|\,|y| = xy$.
Case 2: Let $x < 0$ and $y < 0$. Then $|xy| = -x \cdot -y = xy$. Also, $|x| = -x$ and $|y| = -y$, so $|x|\,|y| = xy$.
Case 3: Let $x \geq 0$ and $y < 0$. Then $xy \leq 0$, so $|xy| = -xy$. Also, $|x| = x$ and $|y| = -y$, so $|x|\,|y| = x \cdot -y = -xy$.
By the three cases, we have proven that $|xy| = |x|\,|y|$. ■

Prove that $\forall x, y \in \mathbb{R},\ |x+y| \leq |x| + |y|$.

Proof

To start, we need to verify that $a \leq |a|$.

Case 1: Let $a \geq 0$. Then $a = |a|$.

Case 2: Let $a < 0$. Then $a < 0 < |a|$.

So we have shown that $a \leq |a|$.

Let $a = xy$. By substitution, we can say that $xy \leq |xy|$. If we multiply both sides by 2, we have that $2xy \leq 2|xy|$. We can add $x^2 + y^2$ to both sides and obtain $x^2 + 2xy + y^2 \leq x^2 + 2|xy| + y^2$. Since the absolute value of a number and the square of a number are both positive, we can say that $x^2 = |x|^2$ and $y^2 = |y|^2$. Also, we have already proven that $|x||y| = |xy|$. Thus, we can obtain the inequality $x^2 + 2xy + y^2 \leq |x|^2 + 2|x||y| + |y|^2$. By factoring both sides, we obtain $(x + y)^2 \leq (|x| + |y|)^2$. By taking the square root of both sides, we obtain $|x + y| \leq |x| + |y|$. ■

3 Contraposition

Prove that if $5z - 7$ is even, then z is odd for all $z \in \mathbb{Z}$.

Proof The contrapositive of the statement is that if z is even, then $5z - 7$ is odd. Hence, let z be an even integer. Then $z = 2m$ for some $m \in \mathbb{Z}$. By substitution, we obtain $5z - 7 = 5(2m) - 7 = 10m - 7 = 10m - 8 + 1 = 2(5m - 4) + 1$. By letting $d = 5m - 4$, we obtain $2d + 1$ for some $d \in \mathbb{Z}$. This implies that $5z - 7$ is an odd integer. Since the contrapositive has been proven true, then we have proven the original statement to be true. ■

Prove that if a^2 is even, then a is even for all $a \in \mathbb{Z}$.

Proof The contrapositive of the statement is that if a is odd, then a^2 is odd. Hence, let a be an odd integer. Then $a = 2m + 1$ for some $m \in \mathbb{Z}$. By substitution, we obtain $a^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$. By closure of integers, we obtain $2d + 1$, $d \in \mathbb{Z}$. This implies that a^2 is an odd integer. Since the contrapositive has been proven true, then we have proven the original statement to be true. ■

Prove that if x is an odd integer, then $\sqrt{2x}$ is not an integer.

Proof The contrapositive of the statement is that if $\sqrt{2x}$ is an integer, then x is an even integer. Hence, let us assume that $\sqrt{2x}$ is an integer. Then $\sqrt{2x} = z$ for some $z \in \mathbb{Z}$. By squaring both sides, we obtain $2x = z^2$. By inspection, we see that z^2 is an even integer. We have already proven that if z^2 then z is even, so z is also an even integer. Hence, we can let $z = 2m$ for some $m \in \mathbb{Z}$. By substitution, we see that $2x = z^2 = (2m)^2 = 4m^2 = 2(2m^2)$. So $2x = 2(2m^2)$. Obviously, we can simplify this to obtain $x = 2m^2$. If we let $p = m^2$, where $p \in \mathbb{Z}$, then we see that $x = 2p$, and hence, x is an even integer. Since the contrapositive has been proven true, then we have proven the original statement to be true. ■

Let x and y be real numbers. Prove that if $x \neq y$ and $x, y \geq 0$, then $x^2 \neq y^2$.

Proof The contrapositive of the statement is that if $x^2 = y^2$, then $x = y$ or x and y are not both non-negative. Let us assume that $x^2 = y^2$. Then $x^2 - y^2 = 0$. By factoring, we have that $(x + y)(x - y) = 0$. Obviously, $x + y = 0$ when $x = -y$, indicating that one of the numbers is negative. Also, $x - y = 0$ when $x = y$. So we have shown that $x^2 = y^2$ implies that $x = y$ or that x or y is negative. Since the contrapositive has been proven true, then we have proven the original statement to be true. ■

Prove that if $x - y$ is an irrational number, then x and y are irrational numbers.

Proof The contrapositive of this statement is that if x and y are rational numbers, then $x - y$ is a rational number. Hence, let x and y be rational numbers. By definition of rational, we can say that $x = \frac{m}{n}$ and $y = \frac{p}{q}$, where $m, n, p, q \in \mathbb{Z}$, $n \neq 0$, and $q \neq 0$. By substitution, we have that $x - y = \frac{m}{n} - \frac{p}{q} = \frac{mq}{nq} - \frac{np}{nq} = \frac{mq - np}{nq}$. By letting $a = mp - np$ and $b = nq$, we obtain $x - y = \frac{a}{b}$ for two integers a and b where $b \neq 0$. Hence, $x - y$ is a rational number. Since the contrapositive has been proven true, then we have proven the original statement to be true. ■

4 Contradiction

Let $a \in \mathbb{Q}$ and let $b \in \mathbb{R} \setminus \mathbb{Q}$. Prove that $a + b \in \mathbb{R} \setminus \mathbb{Q}$.

Proof Suppose for contradiction that $a \in \mathbb{Q}$, $b \in \mathbb{R} \setminus \mathbb{Q}$, and $a + b \in \mathbb{Q}$. Since a is rational, we can say that $a = \frac{l}{m}$ for some $l, m \in \mathbb{Z}$ where $m \neq 0$. Since $a + b$ is rational, we see that $a + b = \frac{j}{k}$ for some $j, k \in \mathbb{Z}$ where $k \neq 0$. Recalling that $a = \frac{l}{m}$, we see that $a + b = \frac{j}{k} \Leftrightarrow \frac{l}{m} + b = \frac{j}{k} \Leftrightarrow b = \frac{j}{k} - \frac{l}{m} = \frac{jm}{km} - \frac{kl}{km} = \frac{jm - kl}{km}$. By letting $c = jm - kl$ and $d = km$, we see that $b = \frac{c}{d}$ for two integers c and d where $d \neq 0$. This implies that b is a rational number. This is a contradiction, since we established earlier that b is an irrational number. This makes the negation false and the original statement true. Hence, we have proven that $a + b \in \mathbb{R} \setminus \mathbb{Q}$. ■

Prove that there is no greatest integer.

Proof Suppose for contradiction that there exists an integer n such that n is the greatest integer. Now, let m be an integer such that $m = n + 1$. Then it is obvious that $m > n$. This contradicts the fact that n is the greatest integer. This makes the negation false and the original statement true. Hence, we have proven that there is no greatest integer. ■