The Binomial Theorem

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Part I
Definition and Proof

1 Theorem

For all real numbers a and b, $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$, where $k, n \in \mathbb{N}$ with $0 \le k \le n$. (We refer to $\binom{n}{k}$ as the *binomial coefficient*, recalling that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.)

2 Proof

To prove the Binomial Theorem, we use the Principle of Mathematical Induction.

Basis Step: Let $n = 0$. By properties of exponents, $(a+b)^0 = 1$. Likewise, $\binom{0}{0} a^{0-0} b^0 = 1 \cdot 1 \cdot 1 = 1$. Since $P(0)$ is true, we may assume that $P(n)$ is true. Thus, we may form the inductive hypothesis that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ for all $n \in \mathbb{N}$.

Inductive Step: [We need to show that $(a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k$.]

By substituting $n+1$, we obtain $(a+b)^{n+1} = (a+b)(a+b)^n = a(a+b)^n + b(a+b)^n$. Now, by our inductive hypothesis, we obtain $a \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k + b \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j = \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^{j+1} =$ $\binom{n}{0} a^{n+1-0} b^0 + \binom{n}{n} a^{n-n} b^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{j=0}^{n-1} \binom{n}{j} a^{n-j} b^{j+1}$. Now, using combinatorial identities, we obtain $a^{n+1} + b^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{j=0}^{n-1} \binom{n}{j} a^{n-j} b^{j+1}$. By letting $j = k-1$, we obtain $a^{n+1} + b^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=1}^{n} \binom{n}{k-1} a^{n-(k-1)} b^k = a^{n+1} + b^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=1}^{n} \binom{n}{k-1} a^{n+1-k} b^k$. By factoring, we obtain $a^{n+1} + b^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1} \right] a^{n+1-k} b^k$. Again, by combinatorial identities, we obtain $\binom{n+1}{0} a^{n+1} + \binom{n+1}{n+1} b^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^{n+1-k} b^k = \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k$. So we have proven that $(a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k$, making the assertion $P(n+1)$ true. Thus, by the principle of mathematical induction, the Binomial Theorem holds for all $n \in \mathbb{N}$. ■

Part II

Applying the Binomial Theorem

3 Binomial Expansions

1. $(a+2b)^4 = \binom{4}{0}a^4(2b)^0 + \binom{4}{1}a^3(2b)^1 + \binom{4}{2}a^2(2b)^2 + \binom{4}{3}a^1(2b)^3 + \binom{4}{4}a^0(2b)^4 = 1 \cdot a^4 \cdot 1 + 4 \cdot a^3 2b + 6 \cdot a^2 \cdot 4b^2 + 4 \cdot a \cdot 8b^3 + 1 \cdot 1 \cdot 16b^4 = a^4 + 8a^3b + 24a^2b^2 + 32ab^3 + 16b^4$

2. $(3a-7b)^3 = \binom{3}{0}(3a)^3(-7b)^0 + \binom{3}{1}(3a)^2(-7b)^1 + \binom{3}{2}(3a)^1(-7b)^2 + \binom{3}{3}(3a)^0(-7b)^3 = 1 \cdot 27a^3 \cdot 1 + 3 \cdot 9a^2 \cdot -7b + 3 \cdot 3a \cdot 49b^2 + 1 \cdot 1 \cdot -343b^3 = 27a^3 - 189a^2b + 441ab^2 - 343b^3$

3. $(a^2+b^2)^3 = \binom{3}{0}(a^2)^3(b^2)^0 + \binom{3}{1}(a^2)^2(b^2)^1 + \binom{3}{2}(a^2)^1(b^2)^2 + \binom{3}{3}(a^2)^0(b^2)^3 = 1 \cdot a^6 \cdot 1 + 3 \cdot a^4 \cdot b^2 + 3 \cdot a^2 \cdot b^4 + 1 \cdot 1 \cdot b^6 = a^6 + 3a^4b^2 + 3a^2b^4 + b^6$

4 Finding Binomial Coefficients of Specific Terms

Find the coefficient of the a^5b^{11} term in the expanded binomial of $(a+b)^{16}$. The power of the binomial is 16, so $n = 16$. Also, we have b^{11}, so $k = 11$, so the binomial coefficient is $\binom{16}{11} = \frac{16!}{11!(16-11)!} = 4368$. Now, a and b both have a coefficient of 1, so the coefficient of the a^5b^{11}term is $4368 \cdot 1 = 4368$.

Find the coefficient of the a^2b^4 term in the expanded binomial of $(2a-3b)^6$. The power of the binomial is 6, so $n = 6$. Also, we have b^4, so $k = 4$, so the binomial coefficient is $\binom{6}{4} = \frac{6!}{4!(6-4)!} = 15$. Now, a has a coefficient of 2 and b has a coefficient of -3, so the coefficient of the a^2b^4 term is $15 \cdot 2 \cdot -3 = -90$.

Find the coefficient of the a^2b^{18} term in the expanded binomial of $(3a^2-5b^2)^{10}$. The power of the binomial is 10, so $n = 10$. Also, we have b^{18}, but since the second monomial is $-5b^2$, $k = \frac{18}{2} = 9$, so the binomial coefficient is $\binom{10}{9} = \frac{10!}{9!(10-9)!} = 10$. Now, a has a coefficient of $3^1 = 3$ and b has a coefficient of $(-5)^9 = -1953125$, so the coefficient of the a^2b^{18} term is $10 \cdot 3 \cdot -1953125 = -58593750$.

5 Writing Combinatorial Sums in Closed Form

We have already seen how the Binomial Theorem can be used to expand expressions of the form $(a+b)^n$. Likewise, we can use the theorem in reverse to put sums involving combinations into a single expression. In order to do so, you must use algebra to manipulate the sum into the form $\sum\limits_{k=0}^{n} \binom{n}{k} a^{n-k}b^k$ and then put it into the form $(a+b)^n$. Often times you must use the multiplicative identity that $1 = 1^k = 1^{n-k}$.

Examples

1. $\sum\limits_{k=0}^{n} \binom{n}{k} = \sum\limits_{k=0}^{n} \binom{n}{k} \cdot 1^k \cdot 1^{n-k} = (1+1)^n = 2^n$

2. $\sum\limits_{k=0}^{n} \binom{n}{k} x^{ak} = \sum\limits_{k=0}^{n} \binom{n}{k} (x^a)^k \cdot 1^{n-k} = (x^a+1)^n$

3. $\sum\limits_{k=0}^{n} (-1)^k \binom{n}{k} 3^{2k-2n} 2^{2k} = \sum\limits_{k=0}^{n} (-1)^k \binom{n}{k} (3^2)^{k-n} (2^2)^k = \sum\limits_{k=0}^{n} \binom{n}{k} (9)^{-(n-k)} (4)^k (-1)^k = \sum\limits_{k=0}^{n} \binom{n}{k} (9^{-1})^{n-k} (-4)^k = \sum\limits_{k=0}^{n} \binom{n}{k} (\frac{1}{9})^{n-k} (-4)^k = (-4+\frac{1}{9})^n = (-\frac{35}{9})^n$.

4. $\sum\limits_{k=0}^{n} (-1)^k \binom{n}{k} 2^{-2k} = \sum\limits_{k=0}^{n} \binom{n}{k} (-1)^k (2^{-2})^k = \sum\limits_{k=0}^{n} \binom{n}{k} (\frac{1}{4} \cdot -1)^k \cdot 1^{n-k} = \sum\limits_{k=0}^{n} \binom{n}{k} (-\frac{1}{4})^k \cdot 1^{n-k} = (1-\frac{1}{4})^n = (\frac{3}{4})^n$