Calculus Proofs

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Theorem 1: The Leibniz Rule

Let f and g be two continuous, differentiable functions. Let $k, n \in \mathbb{N}$ such that $0 \leq k \leq n$. Then $[f(x) \cdot g(x)]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) \cdot g^{(n-k)}(x)$, where n is the order of the derivative.

Proof: Mathematical Induction

Basis Step: Let $n = 1$. By the product rule, we know that $[f(x) \cdot g(x)]' = f(x) \cdot g'(x) + f'(x) \cdot g(x)$. Likewise, $\sum_{k=0}^{1} \binom{1}{k} f^{(k)}(x) \cdot g^{(1-k)}(x) = \binom{1}{0} f^{(0)}(x) \cdot g'(x) + \binom{1}{1} f'(x) \cdot g^{(0)}(x) = f(x) \cdot g'(x) + f'(x) \cdot g(x)$.

Since $P(1)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that$[f(x) \cdot g(x)]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) \cdot g^{(n-k)}(x)$ for all $n \in \mathbb{N}$.

Inductive Step: [We need to show that $[f(x) \cdot g(x)]^{(n+1)} = \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)}(x) \cdot g^{(n+1-k)}(x)$.]

By substituting $n+1$ for n, we obtain $[f(x) \cdot g(x)]^{(n+1)} = \left[[f(x) \cdot g(x)]^{(n)}\right]'$. By our inductive hypothesis, we obtain $\left[\sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) \cdot g^{(n-k)}(x)\right]'$. By the product rule, we obtain $\sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) \cdot g^{(n+1-k)}(x) + f^{(k+1)}(x) \cdot g^{(n-k)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) \cdot g^{(n+1-k)}(x) + \sum_{k=0}^{n} \binom{n}{k} f^{(k+1)}(x) \cdot g^{(n-k)}(x)$

$= \sum_{k=1}^{n} \binom{n}{k} f^{(k)}(x) \cdot g^{(n+1-k)}(x) + \sum_{k=1}^{n} \binom{n}{k-1} f^{(k)}(x) \cdot g^{(n+1-k)}(x) + \binom{n}{0} f(x) \cdot g^{(n+1)}(x) + \binom{n}{n} f^{(n+1)}(x) \cdot g(x) =$ $\binom{n}{0} f(x) \cdot g^{(n+1)}(x) + \binom{n}{n} f^{(n+1)}(x) \cdot g(x) + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1}\right] f^{(k)}(x) \cdot g^{(n+1-k)}(x)$. By Pascal's Rule and other binomial identities, we obtain

$\binom{n+1}{0} f(x) \cdot g^{(n+1)}(x) + \binom{n+1}{n+1} f^{(n+1)}(x) \cdot g(x) + \sum_{k=1}^{n} \binom{n+1}{k} f^{(k)}(x) \cdot g^{(n+1-k)}(x) = \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)}(x) \cdot g^{(n+1-k)}(x)$. So we have proven that $[f(x) \cdot g(x)]^{(n+1)} = \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)}(x) \cdot g^{(n+1-k)}(x)$ as desired, making the assertion $P(n+1)$ true. Thus, by the principle of mathematical induction, we have proven that the Leibniz Rule holds for all derivatives of order $n \geq 1$. ■

Theorem 2: The Power Rule for Derivatives

Let $x \in \mathbb{R}$ and let $n \in \mathbb{N}$. Then $[x^n]' = nx^{n-1}$.

Proof #1: Direct Proof

By defintion of derivative, $[x^n]' = \lim_{h\to 0} \dfrac{(x+h)^n - x^n}{h}$
$= \lim_{h\to 0} \dfrac{\binom{n}{0}x^{n-0}h^0 + \binom{n}{1}x^{n-1}h^1 + ... + \binom{n}{n-1}x^1h^{n-1} + \binom{n}{n}x^0h^{n-0} - x^n}{h} = \lim_{h\to 0} \dfrac{x^n + nx^{n-1}h + ... + nxh^{n-1} + h^n - x^n}{h}$
$= \lim_{h\to 0} \dfrac{nx^{n-1}h + nx^{n-2}h^2 ... + nxh^{n-1}h + h^n}{h} = \lim_{h\to 0} \dfrac{h(nx^{n-1} + nx^{n-2}h ... + nxh^{n-2} + h^{n-1})}{h} = \lim_{h\to 0} nx^{n-1} +$ $nx^{n-2}h ... + nxh^{n-2} + h^{n-1}$. Now, by substituting the limit, we obatin $nx^{n-1} + nx^{n-2}(0) ... + nx0^{n-2} + 0^{n-1} =$ nx^{n-1}. Therefore, the Power Rule holds true for all derivatives. ■

Proof #2: Mathematical Induction

Basis Step: Let $n = 1$. Then $[x^1]' = 1$. Likewise, $1x^{1-1} = 1x^0 = 1 \cdot 1 = 1$. Since $P(1)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that $[x^n]' = nx^{n-1}$ for all $n \in \mathbb{N}$.

Inductive Step: [We need to prove that $[x^{n+1}]' = (n+1)x^n$.]

By substituting $n+1$, we obtain $[x^{n+1}]' = x(x^n)]'$. By using the product rule for deriavtives, we obtain $x[x^n]' + x^n \cdot 1$. By our inductive hypothesis, we obtain $x(nx^{n-1}) + x^n = nx^n + x^n = (n+1)x^n$. So we have proven that $[x^{n+1}]' = (n+1)x^n$ as desired, making the assertion $P(n+1)$ true. Thus, by the principle of mathematical induction, we have proven that the Power Rule holds for all derivatives. ■

Let f and g be two continuous, differentiable functions. Prove that $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$.

Proof

By defintion of derivative, $\frac{d}{dx}[f(x)+g(x)] = \lim_{h\to 0} \dfrac{f(x+h) + g(x+h) - (f(x) + g(x))}{h}$
$= \lim_{h\to 0} \dfrac{f(x+h) + g(x+h) - f(x) - g(x)}{h} = \lim_{h\to 0} \dfrac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$
$= \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \dfrac{g(x+h) - g(x)}{h} = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$. ■