Parseval's Theorem (complex version): Let $f \in L^2[-\pi,\pi]$. Then $\|f(t)\|^2 = \int_{-\pi}^{\pi} |f(t)|^2\, dt = 2\pi \sum_{n=-\infty}^{\infty} |a_n|^2$, where $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\, dt$.

Proof: We will instead prove that $\langle f, g\rangle = 2\pi \sum_{n=-\infty}^{\infty} a_n \overline{b_m}$, where $f = \sum_{n=-\infty}^{\infty} a_n e^{int}$,

$g = \sum_{m=-\infty}^{\infty} b_m e^{imt}$, $f_N = \sum_{n=-N}^{N} a_n e^{int}$, and $g_N = \sum_{m=-N}^{N} a_m e^{imt}$. We will examine the case where $f = g$. First, we need to show that $\langle f_N, g_N\rangle = 2\pi \sum_{n=-N}^{N} a_n \overline{b_m}$. Now, $\langle f_N, g_N\rangle$

$= \langle \sum_{n=-N}^{N} a_n e^{int}, \sum_{m=-N}^{N} b_m e^{imt}\rangle = \sum_{n=-N}^{N} a_n \langle e^{int}, \sum_{m=-N}^{N} b_m e^{imt}\rangle$ by linearity on the first term. By additivity on the second term and conjugate symmetry, we obtain $\sum_{n=-N}^{N} a_n \sum_{m=-N}^{N} \overline{b_m} \langle e^{int}, e^{imt}\rangle$. Now, since $\left\{\frac{e^{int}}{\sqrt{2\pi}}, \frac{e^{imt}}{\sqrt{2\pi}}\right\}$ is an orthonormal set, then $\langle e^{int}, e^{imt}\rangle = \begin{cases} 0, n \neq m \\ 2\pi, n = m \end{cases}$. Since $m = n$, then we obtain $\sum_{n=-N}^{N} a_n \sum_{m=-N}^{N} \overline{b_m}\, 2\pi = \sum_{n=-N}^{N} a_n \overline{b_m} 2\pi$. Hence, $\langle f_N, g_N\rangle = 2\pi \sum_{n=-N}^{N} a_n \overline{b_m}$.

Now we must show that $\langle f_N, g_N\rangle \to \langle f, g\rangle$ as $N \to \infty$. We will do so by showing that $|\langle f_N, g_N\rangle - \langle f, g\rangle| \to 0$ as $N \to \infty$. Now, $|\langle f_N, g_N\rangle - \langle f, g\rangle| = |\langle f_N, g_N\rangle - \langle f, g_N\rangle + \langle f, g_N\rangle - \langle f, g\rangle| = |\langle f_N - f, g_N\rangle + \langle f, g_N - g\rangle|$ by additivity of the first and second terms. By the Schwarz inequality, we obtain $|\langle f_N - f, g_N\rangle + \langle f, g_N - g\rangle| \leq \|f_N - f\|\|g_N\| + \|f\|\|g_N - g\|$. By convergence in the norm, we obtain $|\langle f_N - f, g_N\rangle + \langle f, g_N - g\rangle| \leq 0\|g_N\| + \|f\|0 = 0$. Hence, $|\langle f_N, g_N\rangle - \langle f, g\rangle| \to 0$ as $N \to \infty$. This completes the proof. ■

Define the *convolution* of f and g, denoted $(f * g)(t)$, as $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(x-t)\, dt$. If $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$ and $g(x) = \sum_{n=-\infty}^{\infty} b_n e^{inx}$, then $(f * g)(t) = \sum_{n=-\infty}^{\infty} a_n b_n e^{inx}$.

Proof: From the definition of complex Fourier series, we know that $(f * g)(t) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$, so we need to show that $c_n = a_n b_n$. Now, $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f * g)(t) e^{-inx}\, dx =$

$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(x-t)\, dt\right] e^{-inx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(x-t)\, e^{-inx}\, dt\, dx =$ $\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2\pi} f(t) \int_{-\pi}^{\pi} g(x-t)\, e^{-inx}\, dx\, dt$ by Fubini's Theorem. Now, let $u = x - t$. Then $du = dt$, $x = u + t$, and the limits of integration do not change. Hence, we obtain $\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2\pi} f(t) \int_{-\pi}^{\pi} g(u)\, e^{-in(t+u)} du\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2\pi} f(t) \int_{-\pi}^{\pi} g(u)\, e^{-int} e^{-inu}\, du\, dt =$ $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int} \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} g(u)\, e^{-inu}\, du\right] dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int} b_n\, dt = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\, dt\right] b_n = a_n b_n$. This completes the proof. ■

Riemann-Lebesque Theorem: Let $f(x)$ be a piecewise-continuous function on $[a,b]$. Then $\lim_{k\to\infty}\int_a^b f(t)\cos kt\,dt = \lim_{k\to\infty}\int_a^b f(t)\sin kt\,dt = 0$.

Lemma 1: Let $f(x)$ be a 2π-periodic function and $S_N(x) = a_0 + \sum_{n=1}^{N}[a_n\cos n\pi x + b_n \sin n\pi x]$. Then $S_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\left[\frac{1}{2} + \sum_{k=1}^{N}\cos ku\right]du$.

Proof: By the definitions of Fourier coefficients, we have $S_N(x) = a_0 + \sum_{k=1}^{N}[a_k\cos kx + b_k\sin kx] = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \sum_{k=1}^{N}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt\cos kx + \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt\sin kx\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \sum_{k=1}^{N}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)[\cos kt\cos kx + \sin kt\sin kx]\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \sum_{k=1}^{N}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos k(t-x)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{N} f(t)\cos k(t-x)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{N}\cos k(t-x)\right]dt$.

Now, let $u = t - x$. Then we obtain $\frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(x+u)\left[\frac{1}{2} + \sum_{k=1}^{N}\cos ku\right]du$. Since $f(x)$ is 2π-periodic, we obtain $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\left[\frac{1}{2} + \sum_{k=1}^{N}\cos ku\right]du$ as desired. ■

Lemma 2: For all $k \in \mathbb{R}^+$ and for all $u \in [-\pi,\pi]$, $1/2 + \sum_{k=1}^{N}\cos ku = \begin{cases} N + 1/2, & u = 0 \\ \frac{\sin(N+1/2)u}{2\sin(u/2)}, & u \neq 0 \end{cases}$.

Proof: If $u = 0$, then $\frac{1}{2} + \sum_{k=1}^{N}\cos ku = \frac{1}{2} + \sum_{k=1}^{N}\cos 0 = \frac{1}{2} + \sum_{k=1}^{N} 1 = \frac{1}{2} + N = N + \frac{1}{2}$. Now, if $u \neq 0$, then $\sum_{k=1}^{N}\cos ku\sin\left(\frac{u}{2}\right) = \sum_{k=1}^{N}\frac{1}{2}\left[\sin\left(ku + \frac{u}{2}\right) - \sin\left(ku - \frac{u}{2}\right)\right] = \sum_{k=1}^{N}\frac{1}{2}\left[\sin\left(k + \frac{1}{2}\right)u - \sin\left(k - \frac{1}{2}\right)u\right] = \frac{1}{2}\left[\sin\left(N + \frac{1}{2}\right)u - \sin\left(\frac{u}{2}\right)\right] = \frac{1}{2}\sin\left(N + \frac{1}{2}\right)u - \frac{1}{2}\sin\left(\frac{u}{2}\right)$. Since $\sum_{k=1}^{N}\cos ku\sin\left(\frac{u}{2}\right) = \frac{1}{2}\sin\left(N + \frac{1}{2}\right)u - \frac{1}{2}\sin\left(\frac{u}{2}\right)$, we have $\frac{1}{2}\sin\left(\frac{u}{2}\right) + \sum_{k=1}^{N}\cos ku\sin\left(\frac{u}{2}\right) = \frac{1}{2}\sin\left(N + \frac{1}{2}\right)u \Rightarrow \sin\left(\frac{u}{2}\right)\left[\frac{1}{2} + \sum_{k=1}^{N}\cos ku\right] = \frac{1}{2}\sin\left(N + \frac{1}{2}\right)u \Rightarrow \frac{1}{2} + \sum_{k=1}^{N}\cos ku = \frac{\sin(N+1/2)u}{2\sin(u/2)}$. ■

Lemma 3: Let $P_N(u) = \frac{1}{\pi}\left[\frac{1}{2} + \sum_{k=1}^{N}\cos ku\right]$. Then $\int_{-\pi}^{\pi} P_N(u)\,du = 1$.

Pointwise Convergence Theorem: If $f(x)$ is continuous and 2π-periodic, then for every point x where $f'(x)$ exists, the trigonometric Fourier series, $F(x)$, converges to $f(x)$.

Proof: We need to show that $\lim_{N\to\infty} S_N(x) = f(x)$. We will instead prove that $\lim_{N\to\infty} S_N(x) - f(x) = 0$. By Lemma 1, $\lim_{N\to\infty} S_N(x) - f(x) = \lim_{N\to\infty}\int_{-\pi}^{\pi} P_N(u)f(x+u)du - f(x)\cdot 1$. By Lemma 3, we obtain $\lim_{N\to\infty}\int_{-\pi}^{\pi} P_N(u)f(x+u)du - f(x)\int_{-\pi}^{\pi} P_N(u)\,du = \lim_{N\to\infty}\int_{-\pi}^{\pi} P_N(u)[f(x+u) - f(x)]du$. By Lemma 2, we obtain $\lim_{N\to\infty}\int_{-\pi}^{\pi}\frac{\sin(N+1/2)u}{2\pi\sin(u/2)}[f(x+u) - f(x)]du = \lim_{N\to\infty}\int_{-\pi}^{\pi}\frac{\sin(Nu)\cos(u/2) + \sin(u/2)\cos(Nu)}{2\pi\sin(u/2)}[f(x+u) - f(x)]du = \lim_{N\to\infty}\int_{-\pi}^{\pi}\left[\frac{\sin(Nu)\cos(u/2)}{2\pi\sin(u/2)} + \frac{\cos(Nu)}{2\pi}\right][f(x+u) - f(x)]du =$

$\lim_{N\to\infty}\int_{-\pi}^{\pi}\left[\frac{[f(x+u)-f(x)]\sin(Nu)\cos(u/2)}{2\pi\sin(u/2)}+\frac{[f(x+u)-f(x)]\cos(Nu)}{2\pi}\right]du$. Now, let $g(u)=\frac{[f(x+u)-f(x)]\cos(u/2)}{2\pi\sin(u/2)}$ and $h(u)=\frac{[f(x+u)-f(x)]}{2\pi}$. By substitution and the Riemann-Lebesque Theorem, we obtain $\lim_{N\to\infty}\int_{-\pi}^{\pi}[g(u)\sin Nu+h(u)\cos Nu]du=\lim_{N\to\infty}\int_{-\pi}^{\pi}g(u)\sin Nu\,du+\lim_{N\to\infty}\int_{-\pi}^{\pi}h(u)\cos Nu\,du=0.$ ■