# Fourier Transform Properties/Theorems

## Linearity Property: $\mathcal{F}\{af(t)+bg(t)\} = a\hat{f}(\lambda)+b\hat{g}(\lambda)$

## Time Shifting Property: $\mathcal{F}\{f(t-a)\} = \hat{f}(\lambda)e^{-ia\lambda}$

**Proof:** Using the definition of Fourier Transform, $\mathcal{F}\{f(t-a)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t-a)\,e^{-i\lambda t}\,dt$. Now, let $u = t-a$. Then $du = dt$ and the limits of integration do not change. Hence, $\mathcal{F}\{f(t-a)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{-i\lambda(u+a)}\,du = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{-i\lambda u}\,e^{-i\lambda a}\,du = e^{-i\lambda a}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{-i\lambda u}\,du\right] = e^{-i\lambda a}\hat{f}(\lambda)$. ∎

## Frequency Shifting Property: $\mathcal{F}\{e^{iat}f(t)\} = \hat{f}(\lambda-a)$

**Proof:** Using the definition of Fourier Transform, $\mathcal{F}\{e^{iat}f(t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{iat}\,e^{-i\lambda t}\,dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{iat-i\lambda t}\,dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i(a-\lambda)t}\,dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i(\lambda-a)t}\,dt = \hat{f}(\lambda-a)$. ∎

## Time Scaling Property: $\mathcal{F}\{f(bt)\} = \frac{1}{|b|}\hat{f}\left(\frac{\lambda}{b}\right)$ for any $b \in \mathbb{R}\setminus\{0\}$

**Proof:** We must consider two possible cases. Recall the definition of absolute value, which states that $|b| = \begin{cases} -b, & b < 0 \\ b, & b \geq 0 \end{cases}$ for all $b \in \mathbb{R}$.

*Case 1:* Suppose that $b > 0$. By definition of Fourier Transform, $\mathcal{F}\{f(bt)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(bt)\,e^{-i\lambda t}\,dt$. Now, let $u = bt$. It follows that $t = \frac{u}{b}$ and $dt = \frac{1}{b}du$. Since $b$ is positive, the limits of integration remain the same. Hence, $\mathcal{F}\{f(bt)\} = \mathcal{F}\{f(u)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{-i\lambda(u/b)}\frac{1}{b}du = \frac{1}{b}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{-iu\lambda/b}du = \frac{1}{b}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{-iu(\lambda/b)}du\right] = \frac{1}{b}\hat{f}\left(\frac{\lambda}{b}\right)$. Now, since $b$ is positive, then $|b| = b$, so $\frac{1}{b}\hat{f}\left(\frac{\lambda}{b}\right) = \frac{1}{|b|}\hat{f}\left(\frac{\lambda}{b}\right)$. Thus, $\mathcal{F}\{f(bt)\} = \frac{1}{|b|}\hat{f}\left(\frac{\lambda}{b}\right)$ when $b > 0$.

*Case 2:* Suppose that $b < 0$. By definition of Fourier Transform, $\mathcal{F}\{f(bt)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(bt)\,e^{-i\lambda t}\,dt$. Now, let $v = bt$. It follows that $t = \frac{v}{b}$ and $dt = \frac{1}{b}dt$. Since $b$ is negative, the limits of integration are reversed. Hence, $\mathcal{F}\{f(bt)\} = \mathcal{F}\{f(v)\} = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f(v)\,e^{-i(v/b)\lambda}\frac{1}{b}dv = -\frac{1}{b}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\,e^{-iv\lambda/b}dv = -\frac{1}{b}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\,e^{-iv(\lambda/b)}dv\right] = -\frac{1}{b}\hat{f}\left(\frac{\lambda}{b}\right)$. Now, since $b$ is negative, then $|b| = -b$, so $-\frac{1}{b}\hat{f}\left(\frac{\lambda}{b}\right) = \frac{1}{|b|}\hat{f}\left(\frac{\lambda}{b}\right)$. Thus, $\mathcal{F}\{f(bt)\} = \frac{1}{|b|}\hat{f}\left(\frac{\lambda}{b}\right)$ when $b < 0$.

By the two cases, we have proven that $\mathcal{F}\{f(bt)\} = \frac{1}{|b|}\hat{f}\left(\frac{\lambda}{b}\right)$ for all $b \neq 0$. ∎

## Frequency Scaling Property: $\mathcal{F}\left\{\frac{1}{|b|} f\left(\frac{t}{b}\right)\right\} = \hat{f}(b\lambda)$ for any $b \in \mathbb{R} \setminus \{0\}$

**Proof:** We will start with the right side of the equation, and show that $\mathcal{F}^{-1}\{\hat{f}(b\lambda)\} = \frac{1}{|b|} f\left(\frac{t}{b}\right)$. Now, we must consider two possible cases. (Recall the definition of absolute value, which states that $|b| = \begin{cases} -b, & b < 0 \\ b, & b \geq 0 \end{cases}$ for all $b \in \mathbb{R}$.)

*Case 1:* Suppose that $b > 0$. By definition of Inverse Fourier Transform, $\mathcal{F}^{-1}\{\hat{f}(b\lambda)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(b\lambda)\, e^{i\lambda t}\, d\lambda$. Now, let $u = b\lambda$. It follows that $\lambda = \frac{u}{b}$ and $d\lambda = \frac{1}{b} du$. Since $b$ is positive, the limits of integration remain the same. Hence, $\mathcal{F}^{-1}\{\hat{f}(b\lambda)\} = \mathcal{F}^{-1}\{\hat{f}(u)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(u)\, e^{i(u/b)t} \frac{1}{b} du = \frac{1}{b} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(u)\, e^{iut/b} du = \frac{1}{b}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(u)\, e^{iu(t/b)} du\right] = \frac{1}{b} f\left(\frac{t}{b}\right)$. Now, since $b$ is positive, then $|b| = b$, so $\frac{1}{b} f\left(\frac{t}{b}\right) = \frac{1}{|b|} f\left(\frac{t}{b}\right)$. Thus, $\mathcal{F}^{-1}\{\hat{f}(b\lambda)\} = \frac{1}{|b|} f\left(\frac{t}{b}\right)$ when $b > 0$.

*Case 2:* Suppose that $b < 0$. By definition of Inverse Fourier Transform, $\mathcal{F}^{-1}\{\hat{f}(b\lambda)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(b\lambda)\, e^{i\lambda t}\, d\lambda$. Now, let $v = b\lambda$. It follows that $\lambda = \frac{v}{b}$ and $d\lambda = \frac{1}{b} dv$. Since $b$ is negative, the limits of integration are reversed. Hence, $\mathcal{F}^{-1}\{\hat{f}(b\lambda)\} = \mathcal{F}^{-1}\{\hat{f}(v)\} = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} \hat{f}(v)\, e^{i(v/b)t} \frac{1}{b} du = -\frac{1}{b} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(v)\, e^{ivt/b} dv = -\frac{1}{b}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(v)\, e^{iv(t/b)} dv\right] = -\frac{1}{b} f\left(\frac{t}{b}\right)$. Now, since $b$ is negative, then $|b| = -b$, so $-\frac{1}{b} f\left(\frac{t}{b}\right) = \frac{1}{|b|} f\left(\frac{t}{b}\right)$. Thus, $\mathcal{F}^{-1}\{\hat{f}(b\lambda)\} = \frac{1}{|b|} f\left(\frac{t}{b}\right)$ when $b < 0$.

By the two cases, we have proven that $\mathcal{F}^{-1}\{\hat{f}(b\lambda)\} = \frac{1}{|b|} f\left(\frac{t}{b}\right)$ for all $b \neq 0$. ∎

## Time Differentiation: $\mathcal{F}\left\{\frac{d}{dt} f(t)\right\} = (i\lambda)\hat{f}(\lambda)$

**Proof:** By definition of Fourier Transform, $\mathcal{F}\left\{\frac{d}{dt} f(t)\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\lambda t}\, dt$. To calculate this, we must integrate by parts by letting $u = e^{-i\lambda t}$, $du = -(it)e^{i\lambda t}$, $dv = f'(t)\, dt$, and $v = f(t)$. Hence, $\mathcal{F}\left\{\frac{d}{dt} f(t)\right\} = \frac{1}{\sqrt{2\pi}} \left[f(t) e^{-i\lambda t}\right]_{t=-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, (it) e^{i\lambda t}\, dt = 0 + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, (it) e^{i\lambda t}\, dt$ since $f(t) = 0$ for large $|t|$. Now, $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, (it) e^{i\lambda t}\, dt = \frac{1}{\sqrt{2\pi}} (it) \int_{-\infty}^{\infty} f(t)\, e^{i\lambda t}\, dt = (it)\hat{f}(\lambda)$. This completes the proof. ∎

## Frequency Differentiation: $\mathcal{F}\{(-it) f(t)\} = \frac{d}{d\lambda} \hat{f}(\lambda)$

**Proof:** We will start with the right side, and prove that $\mathcal{F}^{-1}\left\{\frac{d}{d\lambda} \hat{f}(\lambda)\right\} = (-it) f(t)$. By definition of Inverse Fourier Transform, $\mathcal{F}^{-1}\left\{\frac{d}{d\lambda} \hat{f}(\lambda)\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}'(\lambda)\, e^{i\lambda t}\, d\lambda$. To calculate this, we must integrate by parts by letting $u = e^{i\lambda t}$, $du = (it) e^{i\lambda t}\, d\lambda$, $dv = \hat{f}'(\lambda)$, and $v = \hat{f}(\lambda)$. Hence, $\mathcal{F}^{-1}\left\{\frac{d}{d\lambda} \hat{f}(\lambda)\right\} = \frac{1}{\sqrt{2\pi}} \left[\hat{f}(\lambda) e^{i\lambda t}\right]_{\lambda=-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\lambda)\, (it) e^{i\lambda t}\, d\lambda = 0 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\lambda)\, (it) e^{i\lambda t}\, d\lambda$ since $f(\lambda) = 0$ for large

$|\lambda|$. Now, $-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\lambda)\,(it)e^{i\lambda t}\,d\lambda = -(it)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\lambda)\,e^{i\lambda t}\,d\lambda = -(it)f(t)$. Since the equation $\mathcal{F}^{-1}\left\{\frac{d}{d\lambda}\hat{f}(\lambda)\right\} = (-it)f(t)$ is logically equivalent to the equation $\mathcal{F}\{(-it)f(t)\} = \frac{d}{d\lambda}\hat{f}(\lambda)$, then we have proven that $\mathcal{F}\{(-it)f(t)\} = \frac{d}{d\lambda}\hat{f}(\lambda)$. ■

**Convolution Theorem: Define the *convolution* of $f$ and $g$, denoted $(f * g)(t)$, as $\int_{-\infty}^{\infty} f(x)g(t-x)\,dx$. If $f$ and $g$ are piecewise continuous and absolutely integrable, then $\mathcal{F}\{(f*g)(t)\} = \sqrt{2\pi}\hat{f}(\lambda)\hat{g}(\lambda)$.**

**Proof:** By definitions of Fourier Transform and convolution, $\{(f*g)(t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(f*g)(t)\,e^{-i\lambda t}\,dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x)g(t-x)\,dx\right]e^{-i\lambda t}\,dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(t-x)\,e^{-i\lambda t}\,dx\,dt$. By Fubini's Theorem, we obtain $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(t-x)\,e^{-i\lambda t}\,dt\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\int_{-\infty}^{\infty} g(t-x)\,e^{-i\lambda t}\,dt\,dx$. Now, let $u = t - x$. Then $du = dt$ and the limits of integration do not change. Hence, we obtain $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\int_{-\infty}^{\infty} g(u)\,e^{-i\lambda(u+x)}du dx = \int_{-\infty}^{\infty} f(x)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(u)e^{-i\lambda u}\,du\right]e^{-i\lambda x}\,dx = \int_{-\infty}^{\infty} f(x)\hat{g}(\lambda)\,e^{-i\lambda x}\,dx = \hat{g}(\lambda)\int_{-\infty}^{\infty} f(x)\,e^{-i\lambda x}\,dx = \hat{g}(\lambda)\sqrt{2\pi}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-i\lambda x}\,dx\right] = \hat{g}(\lambda)\sqrt{2\pi}\hat{f}(\lambda) = \sqrt{2\pi}\hat{f}(\lambda)\hat{g}(\lambda)$. This completes the proof. ■

**Plancheral Theorem: If $f$ and $g$ are square integrable functions in the vector space $L^2(-\infty,\infty)$, then $< f, g > = < \hat{f}, \hat{g} >$ or $\int_{-\infty}^{\infty} f(t)\,\overline{g(t)}\,dt = \int_{-\infty}^{\infty}\hat{f}(\lambda)\,\overline{\hat{g}(\lambda)}\,d\lambda$.**

**Proof:** By definition of Fourier Transform, and by complex number properties, we have $\int_{-\infty}^{\infty}\hat{f}(\lambda)\,\overline{\hat{g}(\lambda)}\,d\lambda = \int_{-\infty}^{\infty}\hat{f}(\lambda)\overline{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)\,e^{-i\lambda t}\,dt}\,d\lambda = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\lambda)\int_{-\infty}^{\infty}\overline{g(t)}\,e^{i\lambda t}\,dt\,d\lambda$. By Fubini's Theorem, we obtain $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\lambda)\int_{-\infty}^{\infty}\overline{g(t)}\,e^{i\lambda t}\,d\lambda\,dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{g(t)}\left[\int_{-\infty}^{\infty}\hat{f}(\lambda)e^{i\lambda t}\,d\lambda\right]dt = \int_{-\infty}^{\infty}\overline{g(t)}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\lambda)e^{i\lambda t}\,d\lambda\right]dt = \int_{-\infty}^{\infty}\overline{g(t)}f(t)\,dt$. This completes the proof. ■

**Duality Theorem: If $\mathcal{F}\{f(t)\} = \hat{f}(\lambda)$, then $\mathcal{F}\{\hat{f}(t)\} = f(-\lambda)$.**

**Modulation Theorem: If $\mathcal{F}\{f(t)\} = \hat{f}(\lambda)$, then:**

1. **$\mathcal{F}\{f(t)\cos at\} = \frac{1}{2}\left[\hat{f}(\lambda - a) + \hat{f}(\lambda + a)\right]$**
2. **$\mathcal{F}\{f(t)\sin at\} = \frac{1}{2i}\left[\hat{f}(\lambda - a) - \hat{f}(\lambda + a)\right]$.**

**Proof:** By trigonometric identities, time shifting property, and linearity, we have $\mathcal{F}\{f(t)\cos at\} = \mathcal{F}\left\{f(t)\frac{1}{2}\left[e^{iat} + e^{-iat}\right]\right\} = \mathcal{F}\left\{\frac{1}{2}f(t)e^{iat} + \frac{1}{2}f(t)e^{-iat}\right\} = \frac{1}{2}\mathcal{F}\{f(t)e^{iat}\} + \frac{1}{2}\mathcal{F}\{f(t)e^{-iat}\} = \frac{1}{2}\left[\hat{f}(\lambda - a) + \hat{f}(\lambda + a)\right]$. In a similar fashion, we have $\mathcal{F}\{f(t)\sin at\} = \mathcal{F}\left\{f(t)\frac{1}{2i}\left[e^{iat} - e^{-iat}\right]\right\} = \mathcal{F}\left\{\frac{1}{2i}f(t)e^{iat} - \frac{1}{2i}f(t)e^{-iat}\right\} = \frac{1}{2i}\mathcal{F}\{f(t)e^{iat}\} - \frac{1}{2i}\mathcal{F}\{f(t)e^{-iat}\} = \frac{1}{2i}\left[\hat{f}(\lambda - a) - \hat{f}(\lambda + a)\right]$. This completes the proof. ■

## Shannon-Whittaker Sampling Theorem: Let $f$ be a function such that $\hat{f}(\lambda)$ is piecewise differentiable, continuous, and equal to zero for all $\lambda \geq |\Omega|$, where $\Omega$ is a positive frequency. Then $f(t)$ is completely determined by its values at the points $t_j = \frac{j\pi}{\Omega}$, and $f(t) = \sum_{j=-\infty}^{\infty} f\left(\frac{j\pi}{\Omega}\right) \operatorname{sinc}(\Omega t - j\pi)$.

**Proof:** The complex Fourier series for $\hat{f}(\lambda)$ is $\sum_{n=-\infty}^{\infty} C_n e^{in\lambda\pi/\Omega}$, where $C_n = \frac{1}{2\Omega}\int_{-\Omega}^{\Omega} \hat{f}(\lambda) e^{-in\lambda\pi/\Omega}\, d\lambda = \frac{1}{2\Omega}\int_{-\infty}^{\infty} \hat{f}(\lambda) e^{-in\lambda\pi/\Omega} d\lambda$ since $\hat{f}(\lambda) = 0$ outside $(-\Omega, \Omega)$. Now, $C_n = \frac{1}{2\Omega}\int_{-\infty}^{\infty} \hat{f}(\lambda) e^{-in\lambda\pi/\Omega} d\lambda = \frac{\sqrt{2\pi}}{2\Omega}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\lambda) e^{-in\lambda\pi/\Omega} d\lambda = \frac{\sqrt{2\pi}}{2\Omega}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda\left(-\frac{n\pi}{\Omega}\right)} d\lambda\right] = \frac{\sqrt{2\pi}}{2\Omega} f\left(-\frac{n\pi}{\Omega}\right)$. So $\hat{f}(\lambda) = \sum_{n=-\infty}^{\infty} \frac{\sqrt{2\pi}}{2\Omega} f\left(-\frac{n\pi}{\Omega}\right) e^{in(\lambda\pi/\Omega)} = \frac{\sqrt{2\pi}}{2\Omega}\sum_{n=-\infty}^{\infty} f\left(-\frac{n\pi}{\Omega}\right) e^{in\lambda\pi/\Omega}$. By letting $j = -n$, we obtain $\frac{\sqrt{2\pi}}{2\Omega}\sum_{j=-\infty}^{\infty} f\left(\frac{j\pi}{\Omega}\right) e^{-ij\left(\frac{\lambda\pi}{\Omega}\right)}$. Now, by definition of Inverse Fourier Transform, we have $\mathcal{F}^{-1}\{\hat{f}(\lambda)\} =$

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\sqrt{2\pi}}{2\Omega}\sum_{j=-\infty}^{\infty} f\left(\frac{j\pi}{\Omega}\right) e^{-ij(\lambda\pi/\Omega)} e^{i\lambda t}\, d\lambda =$$

$$\frac{1}{\sqrt{2\pi}}\frac{\sqrt{2\pi}}{2\Omega}\sum_{j=-\infty}^{\infty} f\left(\frac{j\pi}{\Omega}\right)\int_{-\infty}^{\infty} e^{-ij(\lambda\pi/\Omega)} e^{i\lambda t}\, d\lambda = \frac{1}{2\Omega}\sum_{j=-\infty}^{\infty} f\left(\frac{j\pi}{\Omega}\right)\int_{-\Omega}^{\Omega} e^{i\lambda(t - j\pi/\Omega)}\, d\lambda =$$

$$\frac{1}{2\Omega}\sum_{j=-\infty}^{\infty} f\left(\frac{j\pi}{\Omega}\right)\left.\frac{e^{i\lambda(t-j\pi/\Omega)}}{i(t-j\pi/\Omega)}\right|_{\lambda=-\Omega}^{\Omega} = \frac{1}{2\Omega}\sum_{j=-\infty}^{\infty} f\left(\frac{j\pi}{\Omega}\right)\frac{2\Omega\sin(\Omega t - j\pi)}{\Omega t - j\pi} = \sum_{j=-\infty}^{\infty} f\left(\frac{j\pi}{\Omega}\right)\operatorname{sinc}(\Omega t - j\pi).$$

This completes the proof. ■

## If $f(t)$ is even, then $\hat{f}(\lambda)$ is real. If $f(t)$ is odd, then $\hat{f}(\lambda)$ is imaginary.

**Proof:** By definition of Fourier transform, $\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{-i\lambda t}\, dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)[\cos\lambda t - i\sin\lambda t]\, dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\cos\lambda t\, dt - \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\sin\lambda t\, dt$. Now, if $f(t)$ is even, then $f(t)\sin\lambda t$ is odd, so $\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\sin\lambda t\, dt = 0$ and $\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\cos\lambda t\, dt$. Similarly, if $f(t)$ is odd, then $f(t)\cos\lambda t$ is odd, so $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\cos\lambda t\, dt = 0$ and $\hat{f}(\lambda) = -\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\sin\lambda t\, dt$. This completes the proof. ■