Proofs involving Normal Subgroups

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Let G be a group. The set $Z(G) = \{x \in G | xg = gx$ for all $g \in G\}$ of all elements that commute with every other element of G is called the *center* of G. Prove that $Z(G)$ is a normal subgroup of G.

Proof The center has been proven to be a subgroup, so we only need to prove normality. Now, let $x \in G$ and let $z \in Z(G)$. By definition of center, $xzx^{-1} = zxx^{-1} = ze = z$. This implies that $xzx^{-1} \in Z(G)$, so the center is a normal subgroup. ■

Let N be a normal subgroup of G with $|N| = 2$. Prove that $N \subseteq Z(G)$.

Proof Since $|N| = 2$, then N has the identity element and a nontrivial element n. Since $n \neq e$ and $o(n)$ divides $|N|$, then $o(n) = 2$. Now, let $x \in G$. Since N is normal, then $xnx^{-1} \in N$. Since $o(xnx^{-1}) = 2$, then $o(xnx^{-1}) = o(n)$, implying that $xnx^{-1} = n$. By multiplying both sides by x on the right, it follows that $xn = nx$. By definition of center, $n \in Z(G)$, and by definition of subset, it follows that $N \subseteq Z(G)$. ■

Theorem 1: Let G/Z be the quotient group of $Z(G)$ in G. If G/Z is cyclic, then G is abelian.

Proof To start, every element in the quotient group G/Z is a right coset Za for some $a \in G$. Since G/Z is cyclic, then $G/Z = \langle Za \rangle$ for some $a \in G$. Now, let $x, y \in G$ so that $Zx, Zy \in G/Z$. It follows that $Zx = (Za)^m = Za^m$ and $Zy = (Za)^n = Za^n$ for some $m, n \in \mathbb{Z}$. From the definition of right cosets, it follows that $x = z_1a^m$ and $y = z_2a^n$ for some $z_1, z_2 \in Z(G)$. By the definition of center and laws of exponents, $xy = z_1a^mz_2a^n = z_1z_2a^ma^n = z_1z_2a^{m+n} = z_1z_2a^{n+m} = z_1z_2a^na^m = z_2z_1a^na^m = z_2a^nz_1a^m = yx$. Hence, G is abelian. ■

Theorem 2: If N is a normal subgroup of G, then the set of right cosets of N forms a group under the coset multiplication given by $NaNb = Nab$.

Proof The subgroup N itself serves as an identity element, since $N = Ne$. It follows that N is closed under coset multiplication since $NeNa = Nea = Na$ and $NaNe = Nae = Na$ for all $a \in G$. The inverse of Na is Na^{-1} since $NaNa^{-1} = Naa^{-1} = Ne = Na^{-1}a = Na^{-1}Na$. Now, to show associativity, let $a, b, c \in G$. Then $(NaNb)Nc = NabNc = N(ab)c = Na(bc) = NaNbc = Na(NbNc)$. Hence N is a group under coset multiplication. ■

Theorem 3: If G is a cyclic group and N is a subgroup of G, then G/N is cyclic.

Proof Let $G = \langle x \rangle$ be a cyclic group with generator x. Now, every cyclic group is abelian, implying that for any $x \in G$ and $n \in N$, $xnx^{-1} = nxx^{-1} = n$. This implies that $xnx^{-1} \in G$, so N is normal in G. Each element of G/N is a right coset Na for some $a \in G$. Since G is cyclic, then $a = x^k$ for some integer k, implying that $Na = Nx^k = (Nx)^k$. Hence, G/N is cyclic with a generator Nx. ■

***Theorem 4*: If G is an abelian group and N is a subgroup of G, then G/N is abelian.**

Proof Let G be an abelian group. By definition of abelian, It follows that for any $x \in G$ and $n \in N$, $xnx^{-1} = nxx^{-1} = n$. This implies that $xnx^{-1} \in G$, so N is normal in G. Each element of G/N is a right coset Na for some $a \in G$. Now, let $b \in G$ so that Nb is also in G/N. Since G is abelian, $NaNb = Nab = Nba = NbNa$. Hence, G/N is abelian. ■

***Theorem* 5: Let N be a normal subgroup of G. If every element of N has finite order and every element of G/N has finite order, then every element of G has finite order.**

Proof Let $x \in G$ so that $Nx \in G/N$. Since Nx has finite order in G/N, then $(Nx)^k = Nx^k = N$ for some integer k. This implies that $x^k \in N$, and by our assumption, x^k has finite order. Now, there exists a positive integer t such that $(x^k)^t = x^{kt} = e$. Hence, every element of G has finite order. ■

***Theorem* 6: The intersection of two normal subgroups is normal.**

Proof Let M and N be normal subgroups of G. Let $a \in M \cap N$ and $g \in G$. By definition of intersection, $a \in M$ and $a \in N$. Since M is normal, we have that $axa^{-1} \in M$. Similarly, $axa^{-1} \in N$. By definition of intersection, it follows that $axa^{-1} \in M \cap N$. Hence, $M \cap N$ is normal in G. ■

Let G be a group, and let N be a normal subgroup of G. Prove that $\forall g \in G$ and $\forall n \in N$ there exist $n_1, n_2 \in N$ such that $gn = n_1 g$ and $ng = gn_2$.

Proof Since N is a normal subgroup, then gng^{-1} and $g^{-1}ng$ are in N. Let $gng^{-1} = n_1$ for some $n_1 \in N$. By multiplying both sides on the right by g, we obtain $gn = n_1 g$. Now, let $g^{-1}ng = n_2$. By multiplying both sides on the left by g, we obtain $ng = gn_2$. ■

Let M and N be normal subgroups of G such that $M \cap N = \{e\}$. Prove that $\forall m \in M$ and $\forall n \in N$, $mn = nm$.

Proof Let $c = mnm^{-1}n^{-1}$. Since N is normal, $mnm^{-1} \in N$. Since the inverse element $n^{-1} \in N$, then $c = (mnm^{-1})n^{-1} \in N$ by closure. Now, M is normal, so $nmn^{-1} \in M$. Since $m \in M$, then $c = m(nm^{-1}n^{-1}) \in M$ by closure. This implies that $c \in M \cap N$, so $c \in \{e\}$. It follows trivially that $c = e$, and by substitution, $mnm^{-1}n^{-1} = e$. Now, by a previous theorem, $e = mnm^{-1}n^{-1} = mn(nm)^{-1}$. By definition of inverse, it follows that $mn = nm$ as desired. ■

Let N be a normal subgroup of index m in G. Prove that $a^m \in N$ for all $a \in G$.

Proof Let $a \in G$. Since $[G : N] = m$, then $|G/N| = m$. From Lagrange's Theorem, it follows that $(Na)^m = Na^m = Ne = N$. Hence, $a^m \in N$. ■

Let N be a normal subgroup of G. Prove that the order of any coset Na in G/N is a divisor of $o(a)$ for any element a in G.

Proof Let $k = o(a)$. Then $(Na)^k = Na^k = Ne = N$. By a previous theorem, $o(Na)$ divides k. ■

Let N be a normal subgroup of G. Prove that G/N is abelian if and only if N contains all elements of the form $aba^{-1}b^{-1}$ for all $a, b \in G$.

Proof:

⇒Suppose that G/N is abelian and that $a, b \in G$. By definition of abelian, $NaNb = NbNa$. By coset multiplication, we obtain $Nab = Nba$. By the criterion for equality of right cosets, it follows that $ab(ba)^{-1} \in$

N. By a previous theorem, $ab(ba)^{-1} = aba^{-1}b^{-1}$. Hence, $aba^{-1}b^{-1} \in N$.
$\Leftarrow$Suppose that $aba^{-1}b^{-1} \in N$ for all $a, b \in G$. By a previous theorem, $aba^{-1}b^{-1} = ab(ba)^{-1}$, so $ab(ba)^{-1} \in N$. By the criterion for equality of right cosets, it follows that $Nab = Nba$. By coset multiplication, we obtain $NaNb = NbNa$. Hence, G/N is abelian.
This completes the proof. ■

Let N be a normal subgroup of G such that $|N| = 7$ and $|G/N| = 20$. Prove that if $x \in G$ and $x^7 = e$, then $x \in N$.

Proof By a previous theorem, since $x^7 = e$, then $o(Nx)$ divides 7. Now, by Lagrange's Theorem, $o(Nx)$ divides $|G/N| = 20$. Since 7 and 20 are relatively prime, then $o(Nx) = 1$, implying that $Nx = N$. It follows that $x \in N$. ■

Let $\phi : G \to G'$ be a group homomorphism. Prove that $\ker(\phi)$ is a normal subgroup.

Proof The kernel of ϕ is nonempty since it obviously contains the identity element e. Now, if $a, b \in \ker(\phi)$, then by definition of homomorphism, $\phi(ab^{-1}) = \phi(a)\phi(b^{-1}) = \phi(a)[\phi(b)]^{-1} = e'e' = e'$. This proves that $\ker(\phi)$ has an inverse element and has closure, so $\ker(\phi)$ is a subgroup. Now, to show that $\ker(\phi)$ is normal, let $k \in \ker(\phi)$ and let $g \in G$. Then $\phi(gkg^{-1}) = \phi(g)\phi(k)\phi(g^{-1}) = \phi(g)e'[\phi(g)]^{-1} = e'$. This shows that $gkg^{-1} \in \ker(\phi)$. Hence, $\ker(\phi)$ is a normal subgroup. ■