Order In The Reals

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1 Supremum and Infimum

1.1 Supremum

1.1.1 Definition

Let S be a nonempty subset of $\mathbb{R}$. The *supremum* is a real number α which satisfies 2 conditions:

- $\forall s \in S$, $s \leq \alpha$.
- If β is a real number such that $s \leq \beta$, then $\alpha \leq \beta$.

The first property states that α is at least as large as every element in S, and the second property states that α is the smallest number greater than any element in S. So $\alpha = \sup(S)$.

1.1.2 Uniqueness of the Supremum

Let S be a nonempty subset of $\mathbb{R}$. If v and w are least upper bounds of S, then $v = w$.

Proof From the hypothesis, both v and w are upper bounds of S. Now, since v is a least upper bound of S, then $v \leq u$ for any upper bound u of S. In particular, w is an upper bound of S, so $v \leq w$. Similarly, w is a least upper bound of S, so $w \leq u$ for any upper bound u of S. In particular, v is an upper bound of S, so $w \leq v$. Since $v \leq w$ and $w \leq v$, we deduce that $v = w$. Hence, the supremum is unique. $\blacksquare$

1.1.3 Characterization of the Supremum

Let S be a nonempty subset of $\mathbb{R}$. If $\alpha = \sup(S)$, then for every $\epsilon > 0$ there exists $s \in S$ such that $s > \alpha - \epsilon$.

Proof Since $\epsilon > 0$, we know that $\alpha - \epsilon < \alpha$. Since α is the least upper bound of S, we can say that $\alpha - \epsilon$ is not an upper bound of S. Thus, we have found an element s in S such that $s > \alpha - \epsilon$.$\blacksquare$

1.2 Infimum

1.2.1 Definition

Let S be a nonempty subset of $\mathbb{R}$. The *infimum* is a real number α which satisfies 2 conditions:

- $\forall s \in S$,$\alpha \leq s$.
- If β is a real number such that $\beta \leq s$, then $\beta \leq \alpha$.

The first property states that α is at least as small as every element in S, and the second property states that α is the greatest number smaller than any element in S. So $\alpha = \inf(S)$.

1.2.2 Uniqueness of the Infimum

Let S be a nonempty subset of $\mathbb{R}$. If v and w are greatest lower bounds of S, then $v = w$.

Proof From the hypothesis, both v and w are lower bounds of S. Now, since v is a greatest lower bound of S, then $l \leq v$ for any lower bound l of S. In particular, w is an lower bound of S, so $w \leq v$. Similarly, w is a greatest lower bound of S, so $l \leq w$ for any lower bound l of S. In particular, v is an lower bound of S, so $v \leq w$. Since $w \leq v$ and $v \leq w$, we deduce that $v = w$. Hence, the infimum is unique. ■

1.2.3 Characterization of the Infimum

Let S be a nonempty subset of $\mathbb{R}$. If $\alpha = \inf(S)$, then for every $\epsilon > 0$ there exists $s \in S$ such that $s < \alpha + \epsilon$.

Proof Since $\epsilon > 0$, we know that $\alpha + \epsilon > \alpha$. Since α is the greatest lower bound of S, we can say that $\alpha + \epsilon$ is not a lower bound of S. Thus, we have found an element s in S such that $s < \alpha + \epsilon$.■

1.3 Proofs

1.3.1 Let $S = \{1 - \frac{1}{n} : n \in Z^+\}$. Prove that $\sup(S) = 1$.

Proof: The first condition is obvious since $1 - \frac{1}{n} < 1$ for all $n \in Z^+$. To prove the second condition, let us suppose for contradiction that $\beta < 1$. Then there exists a positive integer n large enough that $1 - \frac{1}{n} > \beta$. This contradicts the fact that β is an upper bound of S. Therefore, $\sup(S) = 1$. ■

1.3.2 Let $S = \{\frac{1}{n} : n \in Z^+\}$. Prove that $\inf(S) = 0$.

Proof: The first condition is obvious since $0 < \frac{1}{n}$ for all $n \in \mathbb{Z}^+$. To prove the second condition, let us suppose for contradiction that $\beta > \frac{1}{n}$. Then there exists a positive integer n large enough such that $\frac{1}{n} < \beta$. By the archimedean property, we see that $1 < n\beta$. This contradicts the fact that β is a lower bound of S. Hence, $\inf(S) = 0$. ■

1.3.3 Let $S = (0, 1)$. Prove that $\sup(S) = 1$.

Proof: The first condition is obvious since $s < 1$ for all $s \in S$. To prove the second condition, let us suppose for contradiction that $1 > \beta$. Then the real number $\frac{\beta + 1}{2} \in S$. This implies that $0 \leq \beta < \frac{\beta + 1}{2} < 1$. This contradicts the fact that β is an upper bound for S. Therefore, $\sup(S) = 1$. ■

Prove each of the following for all nonempty subsets A and B of $\mathbb{R}$ which are bounded above.

1.3.4 If $A \subseteq B$, then $\sup(A) \leq \sup(B)$.

Proof: Let $x \in A$. By definition of subset, we can say that $x \in B$. By definition of supremum, we have that $x \leq \sup(B)$, so $\sup(B)$ serves as an upper bound for A. By the second property of the supremum, we have that $\sup(A) \leq \sup(B)$. ■

1.3.5 If $x \in \mathbb{R}$ and $B = \{x + a : a \in \mathbb{R}\}$, then $\sup(B) = x + \sup(A)$.

Proof:

$\Rightarrow$Let $b \in B$. Then $b = x + a$ for some $a \in \mathbb{R}$. We know that $a \leq \sup(A)$, so $x + a \leq x + \sup(A)$. Since $x + a = b$, and $x + \sup(A) \geq x + a$, we have that $x + \sup(A)$ is an upper bound for B. Hence, we have that $\sup(B) \leq x + \sup(A)$.
$\Leftarrow$Now let $a \in A$. Since $b = x + a$ for some $a \in A$, we can subtract x from both sides of the equation to obtain $a = -x + b$ for some $b \in B$. Thus, we can define A as $A = \{-x + b : b \in B\}$. Since $b \leq \sup(B)$, we

have that $-x+b \leq -x+\sup(B)$. Since $a=-x+b$, we can say that $-x+\sup(B)$ is an upper bound of A, and so we can say that $\sup(A) \leq -x+\sup(B)$. We can rewrite this to obtain $x+\sup(A) \leq \sup(B)$.
Since we have $x+\sup(A) \leq \sup(B) \leq x+\sup(A)$, we have proven that $\sup(B) = x+\sup(A)$. ■

1.3.6 If $x \in \mathbb{R}$, $x \geq 0$ and $B = \{xa : a \in \mathbb{R}\}$ then $\sup(B) = x\sup(A)$.

Proof:

⇒Let $b \in B$. Then $b = xa$ for some $a \in \mathbb{R}$. We know that $a \leq \sup(A)$, so $xa \leq x\sup(A)$. Since $xa = b$, and $x\sup(A) \geq xa$, we have that $x\sup(A)$ is an upper bound for B. Hence, we have that $\sup(B) \leq x\sup(A)$.
⇐Now let $a \in A$. Since $b = xa$ for some $a \in A$, we can divide x from both sides of the equation to obtain $a = \dfrac{b}{x}$ for some $b \in B$. Thus, we can define A as $A = \{\dfrac{b}{x} : b \in B\}$. Since $b \leq \sup(B)$, we have that $\dfrac{b}{x} \leq \dfrac{\sup(B)}{x}$. Since $a = \dfrac{b}{x}$, we can say that $\dfrac{\sup(B)}{x}$ is an upper bound of A, and so we can say that $\sup(A) \leq \dfrac{\sup(B)}{x}$. We can rewrite this to obtain $x\sup(A) \leq \sup(B)$.
Since we have $x\sup(A) \leq \sup(B) \leq x\sup(A)$, we have proven that $\sup(B) = x\sup(A)$. ■

2 Density

2.1 *Density of Rationals*: Let a and b be two real numbers such that $a < b$. There exists a nonzero rational number c such that $a < c < b$.

Proof:

Without loss of generality, we may assume that $0 < a < b$. From this, we can obtain the inequality $b-a > 0$. By the Archimedean property, there exists a positive integer n that is large enough to where $n(b-a) > 1$. From this, we obtain the inequality $nb - na > 1$.

> *Lemma*: There exists a positive integer m such that $na < m < nb$.
>
> **Proof:** Let m be the minimum of the set $\{k \in \mathbb{Z} : k > na\}$. By definition of minimum, $m > na$. Now, suppose for contradiction that $m \geq nb$. This implies that $m-1 \geq nb-1 > na$. This would imply that $m-1 > na$, and so $m-1 \in \{k \in \mathbb{Z} : k > na\}$. This contradicts the fact that $m = \min\{k \in \mathbb{Z} : k > na\}$. Hence, $m < nb$, and since $m > na$, we have shown that $na < m < nb$ as desired. □

Now, by dividing the inequality $na < m < nb$ by n, we see that $a < \dfrac{m}{n} < b$. Since $m, n \in \mathbb{Z}^+$, then $\dfrac{m}{n}$ is a rational number. By letting $c = \dfrac{m}{n}$, we have found a rational number c such that $a < c < b$ as desired. ■

2.2 Let ϵ be a positive real number. Prove that for every $a \in \mathbb{R}$ there is an rational number b such that $|a-b| < \epsilon$.

Proof:

Since $\epsilon > 0$, then it is obvious that $a - \epsilon < a$. By density of rationals, there exists a rational number b such that $a-\epsilon < b < a$. The first inequality, $a-\epsilon < b$, may be rewritten as $a-b < \epsilon$. The second inequality, $b < a$, may be rewritten by subtracting b from both sides to obtain $0 < a-b$. Since $a-b$ is positive, we can say that $a-b = |a-b|$. Thus, we can rewrite the inequality $a-b < \epsilon$ as $|a-b| < \epsilon$ as desired. ■

2.3 *Density of Irrationals*: For any two real numbers a and b with $a < b$ there exists an irrational number c such that $a < c < b$.

Proof:

Let a and b be two real numbers such that $a < b$. We can divide both by $\sqrt{2}$, a proven irrational number, to obtain $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$. By density of rationals, there exists a nonzero rational number r such that $\frac{a}{\sqrt{2}} < r < \frac{b}{\sqrt{2}}$.

We can manipulate the inequality to obtain $a < r\sqrt{2} < b$. Now, $r\sqrt{2}$ is irrational by Lemma 2.3.1:

> *Lemma 2.3.1:* Let $a \in \mathbb{Q}\backslash\{0\}$ and let $b \in \mathbb{R} \setminus \mathbb{Q}$. Then $ab \in \mathbb{R} \setminus \mathbb{Q}$.
>
> **Proof:** Suppose for contradiction that $ab \in \mathbb{Q}$. Then $ab = r$ for some $r \in \mathbb{Q}$. Since $a \neq 0$, we could say that $b = \frac{r}{a}$. This would imply that b is a rational number, which is a contradiction since we stated that b is not an element of the rational numbers. Hence, $ab \in \mathbb{R} \setminus \mathbb{Q}$. □

Since $r\sqrt{2}$ is now proven to be irrational, then letting $c = r\sqrt{2}$ would imply that c is an irrational number. Thus, we have found an irrational c such that $a < c < b$. ■

2.4 Prove that there exists a rational number between two other rational numbers.

Proof:

Let a and b be two arbitrary rational numbers such that $a < b$. Let $x = \frac{a+b}{2}$. Since $a, b \in \mathbb{Q}$, then $a+b \in \mathbb{Q}$. Since the quotient of rational numbers is rational, we can say that $\frac{a+b}{2} \in \mathbb{Q}$.

Claim #1: [Show that $a < x$.] Since $a < b$, we could say that $a + a < a + b$. So we have that $2a < a + b$, and thus $a < \frac{a+b}{2}$. By substitution, $a < x$.

Claim #2: [Show that $x < b$.] Since $a < b$, we could say that $a + b < b + b$. So we have that $a + b < 2b$, and thus $\frac{a+b}{2} < b$. By substitution, $x < b$.

So we have $a < x < b$, and thus we have found a rational number between two other rational numbers. ■

3 Other Proofs

3.1 *Archimedean Property of Real Numbers:* Let a and b be two positive real numbers. Then there exists a positive integer n such that $a < nb$.

Proof:

Suppose for contradiction that there exists two positive real numbers a and b such that $a \geq nb$ for all $n \in Z^+$. Then we have $\frac{a}{b} \geq n$ for all $n \in Z^+$. This implies that $\frac{a}{b}$ is an upper bound of Z^+. By the completeness axiom of real numbers, it follows that Z^+ has a supremum, which we will call u. From this, it follows that $u - 1$ is not an upper bound of $\mathbb{Z}^+$. So there exists a positive integer m such that $m > u - 1$. Therefore, $m + 1 \in \mathbb{Z}^+$ and $m + 1 > u$. This contradicts the fact that u is the supremum of $\mathbb{Z}^+$. ■

3.2 Prove that $\max\{x, y\} = \frac{|x-y|+x+y}{2}$.

Proof:

Case 1: Suppose $x = y$. Then $\max\{x, y\} = \max\{x, x\} = x$. Likewise, if $x = y$, then $\frac{|x-y|+x+y}{2} = \frac{|x-x|+x+x}{2} = \frac{0+2x}{2} = x$.

Case 2: Suppose $x < y$. Then $\max\{x, y\} = y$. Likewise, if $x < y$, then $\frac{|x-y|+x+y}{2} = \frac{-(x-y)+x+y}{2} = \frac{-x+y+x+y}{2} = \frac{0+2y}{2} = y$.

Case 3: Suppose $x > y$. Then $\max\{x, y\} = x$. Likewise, if $x > y$, then $\frac{|x-y|+x+y}{2} = \frac{x-y+x+y}{2} = \frac{0+2x}{2} = x$.

By the three cases, we have shown that $\max\{x, y\} = \dfrac{|x - y| + x + y}{2}$ as desired. ■