Proofs from Permutations

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Let $\sigma, \tau \in S_n$. Prove that σ and $\tau\sigma\tau^{-1}$ are both even or both odd.

Proof Suppose that σ is the product of s transpositions and that τ is the product of t transpositions. Then τ^{-1} is also the product of t transpositions. This implies that $\tau\sigma\tau^{-1}$ is the product of $s + 2t$ transpositions. Since the sum of an even integer and an odd integer is odd, then s being odd implies that $s + 2t$ is odd. Also, since the sum of two even integers is even, then s being even implies that $s + 2t$ is even. Hence, σ and $\tau\sigma\tau^{-1}$ are both even or both odd. ■

Let $\sigma, \tau \in S_n$. Prove that $\sigma\tau\sigma^{-1}\tau^{-1} \in A_n$.

Proof Suppose that σ is the product of s transpositions and that τ is the product of t transpositions. This implies that σ^{-1} is the product of s transpositions and that τ^{-1} is the product of t transpositions. This implies that $\sigma\tau\sigma^{-1}\tau^{-1}$ is the product of $s + t + s + t$ transpositions. Now, $s + t + s + t = 2s + 2t = 2(s + t)$. This shows that the composition $\sigma\tau\sigma^{-1}\tau^{-1}$ is even. Hence, $\sigma\tau\sigma^{-1}\tau^{-1} \in A_n$. ■

Let α and β be two disjoint cycles in S_n. Prove that the multiplication of the two cycles is abelian.

Proof

In order to prove that $\alpha\beta = \beta\alpha$, we must consider two cases.

Case 1: Suppose that both α and β fix k, i.e. $\alpha(k) = k = \beta(k)$. This implies that $\alpha(\beta(k)) = \alpha(k) = k$ and $\beta(\alpha(k)) = \beta(k) = k$. So $\alpha\beta = \beta\alpha$ when both α and β fix k.

Case 2: Suppose that α moves k and β fixes k. This implies, respectively, that $\alpha(k) = l$ and $\beta(k) = k$. Now, since permutations are injective, we can rule out the possibility that $\alpha(l) = l$. This implies that $\alpha(\beta(k)) = \alpha(k) = l$ and $\beta(\alpha(k)) = \beta(l) = l$. So $\alpha\beta = \beta\alpha$ when α moves k and β fixes k.

From the two cases, it is evident that the multiplication of disjoint cycles is abelian. ■