Mathematical Induction

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Part I
Theorem and Method

1 Principle of Mathematical Induction:

Suppose that for all $n \in \mathbb{N}$, we have an assertion $P(n)$ such that

1. (Basis Step)$P(n_0)$ is true (where n_0 is the minimum of the set containing n)
2. (Inductive Step) If $P(n)$ is true, then $P(n+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof: Suppose that we have a nonempty set A where $A = \{n \in \mathbb{N} : P(n) \text{ is false}\}$. By the Well-Ordering Principle, we can say that A has a unique minimum, which we will call m. Since $m \in A$, then $P(m)$ is false. Now, since $\min(A) = m$, we can say that $m - 1 \notin A$. This implies that $P(m-1)$ is true. By property 2, we deduce that the next term, $P(m)$, is true. This is a contradiction since we found earlier that $P(m)$ is false. Hence, the principle of mathematical induction holds for all $n \in \mathbb{N}$. ■

2 Method

So we know what the Principle of Mathematical Induction states, and we know how to prove the theorem, but how exactly do we use it? Well, we always start out by being given some equality or relation between two sets of numbers that applies to either all natural numbers or all positive integers. There are many uses of this, but it is most often used to prove an inequality between two terms, divisibility of one term by another, or the equality of a sum and a closed formula defined in terms of n. We prove property 1 simply by plugging in the lowest possible integer for n and seeing if a given property is true for that instance. If it is indeed true, then we assume the property holds for all n, i.e., $P(n)$ is true. This is called the *inductive hypothesis.* To prove property 2, we substitute $n+1$ for n and see if a property would hold for the next integer, no matter what n is. When you substitute $n+1$ for n, one side of the inductive hypothesis always appears, and at that point, you must substitute the other side of the hypothesis. From there, you can use simple calculations to prove the $P(n+1)$ assertion to be true.

Part II
Proofs by Mathematical Induction

3 Putting Sums into Closed Form

3.1 Prove that $\sum_{i=0}^{n} i(i!) = (n+1)! - 1$ for all natural numbers n.

Proof:

Basis Step: Let $n = 0$. Then $0(0!) = 0(1) = 0$. Likewise, $(0+1)!-1 = 1!-1 = 1-1 = 0$. Since $P(0)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that $\sum_{i=0}^{n} i(i!) = (n+1)! - 1$ for all $n \in \mathbb{N}$.

Inductive Step: [We must prove that $\sum_{i=0}^{n+1} i(i!) = (n+2)! - 1 \ \forall n \in \mathbb{N}$.]

By substituting $n+1$, we obtain $\sum_{i=0}^{n+1} i(i!) = \sum_{i=0}^{n} i(i!) + (n+1)(n+1)!$. By our inductive hypothesis, we obtain
$(n+1)!-1+(n+1)(n+1)! = [(n+1)+1](n+1)!-1 = (n+2)(n+1)!-1 = (n+2)!-1 = (n+2)!-1 = (n+2)!-1$.
So we have proven that $\sum_{i=0}^{n+1} i(i!) = (n+2)! - 1$, thus proving the assertion $P(n+1)$ to be true. Hence, by the principle of mathematical induction, we have proven that $\sum_{i=0}^{n} i(i!) = (n+1)! - 1$ for all natural numbers. ■

3.2 Prove that $\sum_{i=0}^{n} \frac{i}{(i+1)!} = \frac{(n+1)! - 1}{(n+1)!}$ for all natural numbers n.

Proof:

Basis Step: Let $n = 0$. Then $\frac{0}{(0+1)!} = \frac{0}{1!} = \frac{0}{1} = 0$. Likewise, $\frac{(0+1)! - 1}{(0+1)!} = \frac{1! - 1}{1!} = \frac{0}{1} = 0$. Since $P(0)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that $\sum_{i=0}^{n} \frac{i}{(i+1)!} = \frac{(n+1)! - 1}{(n+1)!}$ for all $n \in \mathbb{N}$.

Inductive Step: [We must prove that $\sum_{i=0}^{n+1} \frac{i}{(i+1)!} = \frac{(n+2)! - 1}{(n+2)!} \ \forall n \in \mathbb{N}$.]

By substituting $n+1$, we obtain $\sum_{i=0}^{n+1} \frac{i}{(i+1)!} = \sum_{i=0}^{n} \frac{i}{(i+1)!} + \frac{n+1}{(n+2)!}$. By our inductive hypothesis, we obtain $\frac{(n+1)! - 1}{(n+1)!} + \frac{n+1}{(n+2)!} = \frac{(n+1)! - 1}{(n+1)!} \cdot \frac{n+2}{n+2} + \frac{n+1}{(n+2)!} = \frac{(n+2)! - (n+2)}{(n+2)!} + \frac{n+1}{(n+2)!} = \frac{(n+2)! - n - 2 + n + 1}{(n+2)!} = \frac{(n+2)! - 1}{(n+2)!}$. So we have proven that $\sum_{i=0}^{n+1} \frac{i}{(i+1)!} = \frac{(n+2)! - 1}{(n+2)!}$, thus proving the assertion $P(n+1)$ to be true. Hence, by the principle of mathematical induction, we have proven that $\sum_{i=0}^{n} \frac{i}{(i+1)!} = \frac{(n+1)! - 1}{(n+1)!}$ for all natural numbers. ■

3.3 Prove that $\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$ for all real numbers $r \neq 1$ and all natural numbers n.

Proof:

Basis Step: Let $n = 0$. Then $r^0 = 1$. Likewise, $\frac{r^{0+1}-1}{r-1} = \frac{r^1-1}{r-1} = \frac{r-1}{r-1} = 1$. Since $P(0)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that $\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$ for all $n \in \mathbb{N}$.

Inductive Step: [We must prove that $\sum_{i=0}^{n+1} r^i = \frac{r^{n+2}-1}{r-1}$ for all $n \in \mathbb{N}$.]

By substituting $n+1$, we obtain $\sum_{i=0}^{n+1} r^i = \sum_{i=0}^{n} r^i + r^{n+1}$. By our inductive hypothesis, we obtain $\frac{r^{n+1}-1}{r-1} + r^{n+1} = \frac{r^{n+1}-1}{r-1} + \frac{(r-1)r^{n+1}}{r-1} = \frac{r^{n+1}-1+(r-1)r^{n+1}}{r-1} = \frac{r(r^{n+1})-1}{r-1} = \frac{r^{n+2}-1}{r-1}$. So we have proven that $\sum_{i=0}^{n+1} r^i = \frac{r^{n+2}-1}{r-1}$, thus proving the assertion $P(n+1)$ to be true. Hence, by the principle of mathematical induction, we have proven that $\sum_{i=0}^{n} i(i!) = (n+1)! - 1$ for all natural numbers. ■

3.4 Prove that $\sum_{i=0}^{n} ai = \frac{a[n(n+1)]}{2}$ for all real numbers a and all natural numbers n.

Proof:

Basis Step: Let $n = 0$. Then $a(0) = 0$. Likewise, $\frac{a[0(0+1)]}{2} = \frac{a[0(1)]}{2} = \frac{a(0)}{2} = \frac{0}{2} = 0$. Since $P(0)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that $\sum_{i=0}^{n} ai = \frac{a[n(n+1)]}{2}$ for all $n \in \mathbb{N}$.

Inductive Step: [We must prove that $\sum_{i=0}^{n+1} ai = \frac{a[(n+1)(n+2)]}{2}$ for all $n \in \mathbb{N}$.]

By substituting $n+1$, we obtain $\sum_{i=0}^{n+1} ai = \sum_{i=0}^{n} ai + a(n+1)$. By our inductive hypothesis, we obtain $\frac{a[n(n+1)]}{2} + a(n+1) = \frac{an(n+1)}{2} + \frac{2a(n+1)}{2} = \frac{an(n+1)+2a(n+1)}{2} = \frac{a[n(n+1)+2(n+1)]}{2} = \frac{a[(n+1)(n+2)]}{2}$. So we have proven that $\sum_{i=0}^{n+1} ai = \frac{a[(n+1)(n+2)]}{2}$, thus proving the assertion $P(n+1)$ to be true. Hence, by the principle of mathematical induction, we have proven that $\sum_{i=0}^{n} ai = \frac{a[n(n+1)]}{2}$ for all natural numbers. ■

4 Putting Recursive Sequences into Closed Form

4.1 Let $a_1 = 0$, $a_2 = 1$, and $a_n = 3a_{n-1} - 2a_{n-2}$. Prove that $a_n = 2^{n-1} - 1$ for all positive integers $n \geq 3$.

Proof:

Basis Step: Let $n = 3$. Then $a_3 = 3a_{3-1} - 2a_{3-2} = 3a_2 - 2a_1 = 3(1) - 2(0) = 3$. Likewise, $2^{3-1} - 1 = 2^2 - 1 = 4 - 1 = 3$. Since $P(3)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that $a_n = 2^{n-1} - 1$ for all positive integers $n \geq 3$.

Inductive Step: [We must show that $a_{n+1} = 2^n - 1$.]

By substituting $n+1$, we obtain $a_{n+1} = 3a_{n+1-1} - 2a_{n+1-2} = 3a_n - 2a_{n-1}$. By inductive hypothesis, we obtain $3(2^{n-1} - 1) - 2(2^{n-2} - 1) = 3(2^{n-1}) - 3 - 2^{n-1} + 2 = 2(2^{n-1}) - 1 = 2^n - 1$. So have shown that $a_{n+1} = 2^n - 1$ as desired, making the assertion $P(n+1)$ true. Thus, by the principle of mathematical induction, we have proven that $a_n = 2^{n-1} - 1$ for all for all positive integers $n \geq 3$. ■

4.2 Let $a_1 = 5$ and $a_n = a_{n-1} + 2$. Prove that $a_n = 2n + 3$ for all positive integers $n \geq 2$.

Proof:

Basis Step: Let $n = 2$. Then $a_2 = 5 + 2 = 7$. Likewise, $2(2) + 3 = 4 + 3 = 7$. Since $P(2)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that $a_n = 2n + 3$ for all $n \in \mathbb{N}$.

Inductive Step: [We must show that $a_{n+1} = 2(n+1) + 3$.]

By substituting $n+1$, we obtain $a_{n+1} = a_n + 2$. By our inductive hypothesis, we obtain $2n + 3 + 2 = 2n + 2 + 3 = 2(n+1) + 3$. So we have shown that $a_{n+1} = 2(n+1) + 3$ as desired, making the assertion $P(n+1)$ true. Thus, by the principle of mathematical induction, we have proven that $a_n = 2n + 3$ for all positive integers $n \geq 2$. ■

4.3 The triangular number, denoted Tr_n, can be found using a recursive formula, where $Tr_1 = 1$, and $Tr_n = Tr_{n-1} + n$. Prove that $Tr_n = \dfrac{n(n+1)}{2}$ for all for all positive integers $n \geq 1$.

Proof:

Basis Step: Let $n = 1$. Then $Tr_1 = 1$. Likewise, $\frac{1(1+1)}{2} = \frac{1(2)}{2} = \frac{2}{2} = 1$. Since $P(1)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that $Tr_n = \frac{n(n+1)}{2}$ for all $n \geq 1$.

Inductive Step: [We must show that $Tr_{n+1} = \frac{(n+1)(n+2)}{2}$.]

By substituting $n+1$, we obtain $Tr_{n+1} = Tr_n + (n+1)$. By inductive hypothesis, we obtain $\frac{n(n+1)}{2} + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$. So we have proven that $Tr_{n+1} = \frac{(n+1)(n+2)}{2}$ as desired, making the assertion $P(n+1)$ true. Thus, the principle of mathematical induction, we have proven that $Tr_n = \frac{n(n+1)}{2}$ for all $n \geq 1$. ■

4.4 The tetrahedral number, denoted Te_n, can be found using a recursive formula, where $Te_1 = 1$, and $Te_n = Te_{n-1} + \dfrac{n(n+1)}{2}$. Prove that $Te_n = \dfrac{n(n+1)(n+2)}{6}$ for all for all positive integers $n \geq 1$.

Proof:

Basis Step: Let $n = 1$. Then $Te_1 = 1$. Likewise, $\frac{1(1+1)(1+2)}{6} = \frac{1(2)(3)}{6} = \frac{6}{6} = 1$. Since $P(1)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that $Te_n = \frac{n(n+1)(n+2)}{6}$ for all $n \geq 1$.

Inductive Step: [We must show that $Te_{n+1} = \frac{(n+1)(n+2)(n+3)}{6}$.]

By substituting $n+1$, we have that $Te_{n+1} = Te_n + \frac{(n+1)(n+1+1)}{2} = Te_n + \frac{(n+1)(n+2)}{2}$. By inductive hypothesis, we obtain $\frac{n(n+1)(n+2)}{6} + \frac{(n+1)(n+2)}{2} = \frac{n(n+1)(n+2)}{6} + \frac{3(n+1)(n+2)}{6} = \frac{n(n+1)(n+2)+3(n+1)(n+2)}{6} = \frac{(n+1)(n+2)(n+3)}{6}$. So we have proven that $Te_{n+1} = \frac{(n+1)(n+2)(n+3)}{6}$ as desired, making the assertion $P(n+1)$ true. Thus, by the principle of mathematical induction, we have proven that $Te_n = \frac{n(n+1)(n+2)}{6}$ for all $n \geq 1$. ■

5 Recursive Identities

5.1 Let $a_n$and b_n be two arbitrary sequences. Prove that $\sum_{i=0}^{n}(a_i + b_i) = \sum_{i=0}^{n} a_i + \sum_{i=0}^{n} b_i$ for all $n \in \mathbb{N}$.

Proof:

Basis Step: Let $n = 0$. Then $\sum_{i=0}^{0}(a_i + b_i) = a_0 + b_0$. Likewise, $\sum_{i=0}^{0} a_i + \sum_{i=0}^{0} b_i = a_0 + b_0$. Since $P(0)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that $\sum_{i=0}^{n}(a_i + b_i) = \sum_{i=0}^{n} a_i + \sum_{i=0}^{n} b_i$ for all $n \in \mathbb{N}$.

Inductive Step: [We must prove that $\sum_{i=0}^{n+1}(a_i + b_i) = \sum_{i=0}^{n+1} a_i + \sum_{i=0}^{n+1} b_i$ for all $n \in \mathbb{N}$.]

By substituting $n+1$, we obtain $\sum_{i=0}^{n+1}(a_i + b_i) = \sum_{i=0}^{n}(a_i + b_i) + a_{n+1} + b_{n+1}$. By our inductive hypothesis, we obtain $\sum_{i=0}^{n} a_i + \sum_{i=0}^{n} b_i + a_{n+1} + b_{n+1} = \sum_{i=0}^{n} a_i + a_{n+1} + \sum_{i=0}^{n} b_i + b_{n+1} = \sum_{i=0}^{n+1} a_i + \sum_{i=0}^{n+1} b_i$. So we have proven that $\sum_{i=0}^{n+1}(a_i + b_i) = \sum_{i=0}^{n+1} a_i + \sum_{i=0}^{n+1} b_i$, thus proving the assertion $P(n+1)$ to be true. Hence, by the principle of mathematical induction, we have proven that $\sum_{i=0}^{n} a_i + b_i = \sum_{i=0}^{n} a_i + \sum_{i=0}^{n} b_i$ for all natural numbers. ■

5.2 Let $a_n$be an arbitrary sequence. Prove that $c\sum_{i=0}^{n} a_i = \sum_{i=0}^{n} c(a_i)$ for all $n \in \mathbb{N}$.

Proof:

Basis Step: Let $n = 0$. Then $c\sum_{i=0}^{0} a_i = c(a_0)$. Likewise, $\sum_{i=0}^{0} c(a_i) = c(a_0)$. Since $P(0)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that $c\sum_{i=0}^{n} a_i = \sum_{i=0}^{n} c(a_i)$ for all $n \in \mathbb{N}$.

Inductive Step: [We must prove that $c\sum_{i=0}^{n+1} a_i = \sum_{i=0}^{n+1} c(a_i)$ for all $n \in \mathbb{N}$.]

By substituting $n+1$, we obtain $c\sum_{i=0}^{n+1} a_i = c\sum_{i=0}^{n} a_i + c(a_{n+1})$. By our inductive hypothesis, we obtain $\sum_{i=0}^{n} c(a_i) + c(a_{n+1}) = \sum_{i=0}^{n+1} c(a_i)$. So we have proven that $c\sum_{i=0}^{n+1} a_i = \sum_{i=0}^{n+1} c(a_i)$, thus proving the assertion

$P(n+1)$ to be true. Hence, by the principle of mathematical induction, we have proven that $c\sum_{i=0}^{n} a_i = \sum_{i=0}^{n} c(a_i)$ for all natural numbers. ■

6 Divisibility

6.1 Prove that $6|n^3 - n$ for all $n \in \mathbb{N}$.

Proof:

Basis Step: Let $n = 0$. Then $0^3 - 0 = 0 - 0 = 0 = 6(0)$. Since $P(0)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that $\forall n \in \mathbb{N}$, $n^3 - n = 6d$ for some $d \in \mathbb{Z}$.

Inductive Step: [We must prove that $(n+1)^3 - (n+1) = 6z$ for some $z \in \mathbb{Z}$.]

By substituting $n+1$, we obtain$(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - n - 1 = n^3 - n + 3n^2 + 3n$. By factoring, we obtain $n^3 - n + 3(n^2 + n) = n^3 - n + 3[n(n+1)]$. Now, we have a product of two consecutive integers, n and $n+1$ The product of two consecutive integers is always even, so we can let $n(n+1) = 2k$ for some $k \in \mathbb{Z}$ by the definition of even integer. Hence, $n^3 - n + 3[n(n+1)] = n^3 - n + 3(2k) = n^3 - n + 6k$. By our inductive hypothesis, we have that $n^3 - n + 6k = 6d + 6k = 6(d+k)$. By letting $z = d + k$, we have shown that $(n+1)^3 - (n+1) = 6z$ for some $z \in \mathbb{Z}$ as desired. We have proven the assertion $P(n+1)$ to be true, so by the principle of mathematical induction, we have proven that $6|n^3 - n$ for all $n \in \mathbb{N}$. ■

6.2 Prove that $3|n^3 - 7n + 3$ for all $n \in \mathbb{N}$.

Proof:

Basis Step: Let $n = 0$. Then $0^3 - 2(0) + 3 = 0 - 0 + 3 = 3 = 3(1)$. Since $P(0)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that $\forall n \in \mathbb{N}$, $n^3 - 7n + 3 = 3k$ for some $k \in \mathbb{Z}$.

Inductive Step: [We must prove that $(n+1)^3 - 7(n+1) + 3 = 3z$ for some $z \in \mathbb{Z}$.]

By substituting $n+1$, we obtain$(n+1)^3 - 7(n+1) + 3 = n^3 + 3n^2 + 3n + 1 - 7n - 7 + 3 = (n^3 - 7n + 3) + 3n^2 + 3n - 6$. By our inductive hypothesis, we obtain $3k + 3n^2 + 3n - 6 = 3(k + n^2 + n - 2)$. By letting $z = k + n^2 + n - 2$, we have shown that $(n+1)^3 - 7(n+1) + 3 = 3z$ for some $z \in \mathbb{Z}$ as desired. We have proven the assertion $P(n+1)$ to be true, so by the principle of mathematical induction, we have proven that $3|n^3 - 7n + 3$ for all $n \in \mathbb{N}$. ■

6.3 Prove that for all non-zero integers a, and for all $n \in \mathbb{N}$, $a|(a+1)^n - 1$.

Proof:

Basis Step: Let $n = 0$. Then $(a+1)^0 - 1 = 1 - 1 = 0 = a(0)$. Since $P(0)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that $\forall n \in \mathbb{N}$, $(a+1)^n - 1 = ak$ for some $k \in \mathbb{Z}$.

Inductive Step: [We must prove that $(a+1)^{n+1} - 1 = az$ for some $z \in \mathbb{Z}$.]

By substituting $n+1$, we obtain $(a+1)^{n+1} - 1 = (a+1)(a+1)^n - 1 = a(a+1)^n + (a+1)^n - 1$. By our inductive hypothesis. we obtain $a(a+1)^n + ak = a[(a+1)^n + k]$. By letting $z = (a+1)^n + k$, we have shown that $(a+1)^{n+1} - 1 = az$ for some $z \in \mathbb{Z}$ as desired. We have proven the assertion $P(n+1)$ to be true, so by the principle of mathematical induction, we have proven that for all non-zero integers a, and for all $n \in \mathbb{N}$, $a|(a+1)^n - 1$. ■

6.4 Prove that $a-b|a^n-b^n$ for all $n \in \mathbb{N}$, where $a>b$.

Proof:

Basis Step: Let $n=0$. Then $a^0-b^0=1-1=0=(a-b)(0)$. Since $P(0)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that $\forall n \in \mathbb{N}$, $a^n-b^n=(a-b)k$ for some $k \in \mathbb{Z}$.

Inductive Step: [We must prove that $a^{n+1}-b^{n+1}=(a-b)z$ for some $z \in \mathbb{Z}$.]

By substituting $n+1$, we obtain
$a^{n+1}-b^{n+1}=a(a^n)-b(b^n)=(a-b+b)(a^n)-b(b^n)=(a-b)a^n+b(a^n)-b(b^n)=(a-b)a^n+b(a^n-b^n)$. By our inductive hypothesis, we obtain $=(a-b)a^n+b[(a-b)k]=(a-b)a^n+(a-b)bk=(a-b)(a^n+bk)$. By letting $z=a^n+bk$, we have shown that $a^{n+1}-b^{n+1}=(a-b)k$ for some $k \in \mathbb{Z}$ as desired. We have proven the assertion $P(n+1)$ to be true, so by the principle of mathematical induction, we have proven that $a-b|a^n-b^n$ for all $n \in \mathbb{N}$. ■

6.5 Prove that $a+b|a^{2n+1}+b^{2n+1}$ for all $n \in \mathbb{N}$.

Proof:

Basis Step: Let $n=0$. Then $a^{2(0)+1}+b^{2(0)+1}=a^1+b^1=a+b=(a+b)(1)$. Since $P(0)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that $\forall n \in \mathbb{N}$, $a^{2n+1}+b^{2n+1}=(a+b)k$ for some $k \in \mathbb{Z}$.

Inductive Step: [We must prove that $a^{2(n+1)+1}+b^{2(n+1)+1}=(a+b)z$ for some $z \in \mathbb{Z}$.]

By substituting $n+1$, we obtain
$a^{2(n+1)+1}+b^{2(n+1)+1}=a^{2n+3}+b^{2n+3}=a^2(a^{2n+1})+b^2(b^{2n+1})=(a^2-b^2+b^2)(a^{2n+1})+b^2(b^{2n+1})$. By distributing and factoring terms, we obtain
$a^2(a^{2n+1})-b^2(a^{2n+1})+b^2(a^{2n+1})+b^2(b^{2n+1})=(a^2-b^2)(a^{2n+1})+b^2(a^{2n+1}+b^{2n+1})$. By our inductive hypothesis, we obtain
$(a^2-b^2)(a^{2n+1})+b^2[(a+b)k]=(a+b)(a-b)(a^{2n+1})+(a+b)b^2k=(a+b)[(a-b)(a^{2n+1})+b^2k]$. By letting $z=(a-b)(a^{2n+1})+b^2k$, we have shown that $a^{2(n+1)+1}+b^{2(n+1)+1}=(a+b)z$ for some $z \in \mathbb{Z}$ as desired. We have proven the assertion $P(n+1)$ to be true, so by the principle of mathematical induction, we have proven that $a+b|a^{2n+1}+b^{2n+1}$ for all $n \in \mathbb{N}$. ■

6.6 Prove that $a-1|a^{2n}-1$ for all $n \in \mathbb{N}$.

Proof:

Basis Step: Let $n=0$. Then $a^{2(0)}-1=a^0-1=1-1=0=(a-1)(0)$. Since $P(0)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that $\forall n \in \mathbb{N}$, $a^{2n}-1=(a-1)k$ for some $k \in \mathbb{Z}$.

Inductive Step: [We must prove that $a^{2(n+1)}-1=(a-1)z$ for some $z \in \mathbb{Z}$.]

By substituting $n+1$, we obtain
$a^{2(n+1)}-1=a^{2n+2}-1=a^2a^{2n}-1=(a^2-1+1)a^{2n}-1=(a^2-1)a^{2n}+a^{2n}-1$. By our inductive hypothesis, we obtain that $(a^2-1)a^{2n}+(a-1)k=(a-1)(a+1)a^{2n}+(a-1)k=(a-1)[(a+1)a^{2n}+k]$. By letting $z=(a+1)a^{2n}+k$, we have shown that $a^{2(n+1)}-1=(a-1)z$ for some $z \in \mathbb{Z}$ as desired. We have proven the assertion $P(n+1)$ to be true, so by the principle of mathematical induction, we have proven that $a-1|a^{2n}-1$ for all $n \in \mathbb{N}$. ■

7 Inequalities

7.1 Prove that for all positive integers $n \geq 1$, $3^n + 2 \geq 3n$.

Proof:

Basis Step: Let $n = 1$. Then $3^1 + 2 \geq 3(1) \Leftrightarrow 3 + 2 \geq 3 \Leftrightarrow 5 \geq 3$. Since $P(1)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that for all positive integers $n \geq 1$, $3^n + 2 \geq 3n$.

Inductive Step: [We must show that $3^{n+1} + 2 \geq 3(n+1)$.]

By substituting $n+1$ for n on the right side of the inequality, we obtain $3(n+1) = 3n + 3$. By our inductive hypothesis, we assume that $3^n + 2 + 3 \geq 3n + 3 = 3(n+1)$. Now we work on the left side. Since $2(3^n) \geq 3$ for all $n \geq 1$, we obtain $3^n + 2 + 2(3^n) \geq 3^n + 2 + 3 \Leftrightarrow 3^{n+1} + 2 = 3^n + 2 + 2(3^n) \geq 3^n + 2 + 3$. By combining the two inequalities, we obtain $3^{n+1} + 2 = 3^n + 2 + 2(3^n) \geq 3^n + 2 + 3 \geq 3n + 3 = 3(n+1)$. This proves that $3^{n+1} + 2 \geq 3(n+1)$ as desired, making the assertion $P(n+1)$ true. Thus, by the principle of mathematical induction, we have proven that $3^n + 2 \geq 3n$ for all positive integers $n \geq 1$. ■

7.2 Prove that for all positive integers $n \geq 2$, $6^n \geq 5^n + 9$.

Proof:

Basis Step: Let $n = 2$. Then $6^2 \geq 5^2 + 9 \Leftrightarrow 36 \geq 25 + 9 \Leftrightarrow 36 \geq 34$. Since $P(2)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that for all positive integers $n \geq 2$, $6^n \geq 5^n + 9$.

Inductive Step: [We must show that $6^{n+1} \geq 5^{n+1} + 9$.]

By substituting $n+1$ for n on the right side of the inequality, we obtain $6^{n+1} = 6(6^n)$. By our inductive hypothesis, we assume that $6^{n+1} = 6(6^n) \geq 6(5^n + 9)$. Now we work on the right side, which we can simplify to obtain $6(5^n) + 54$. It is obvious that $6(5^n) + 54 \geq 5(5^n) + 9$ for all $n \geq 2$, so $6(5^n + 9) \geq 5(5^n) + 9 = 5^{n+1} + 9$. By combining the two inequalities, we obtain $6^{n+1} = 6(6^n) \geq 6(5^n + 9) \geq 5(5^n) + 9 = 5^{n+1} + 9$. This proves that $6^{n+1} \geq 5^{n+1} + 9$ as desired, making the assertion $P(n+1)$ true. Thus, by the principle of mathematical induction, we have proven that $6^n \geq 5^n + 9$ for all positive integers $n \geq 2$. ■

7.3 Prove that for all positive integers $n \geq 3$, $2n + 1 \leq n^2$.

Proof:

Basis Step: Let $n = 3$. Then $2(3) + 1 \leq (3)^2 \Leftrightarrow 6 + 1 \leq 9 \Leftrightarrow 7 \leq 9$. Since $P(3)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that for all positive integers $n \geq 3$, $2n + 1 \leq n^2$.

Inductive Step: [We must show that $2(n+1) + 1 \leq (n+1)^2$.]

By substituting $n+1$ for n on the left side of the inequality, we obtain $2(n+1) + 1 = 2n + 2 + 1$. By our inductive hypothesis, we assume that $2(n+1) + 1 = 2n + 2 + 1 \leq n^2 + 2$. Now we work on the right side. Since $2 \leq 2n + 1$ for all $n \geq 3$, then we obtain the inequality $n^2 + 2 \leq n^2 + 2n + 1 = (n+1)^2$. By combining the two inequalities, we obtain $2(n+1) + 1 = 2n + 2 + 1 \leq n^2 + 2 \leq n^2 + 2n + 1 = (n+1)^2$. This proves that $2(n+1) + 1 \leq (n+1)^2$ as desired, making the assertion $P(n+1)$ true. Thus, by the principle of mathematical induction, we have proven that $2n + 1 \leq n^2$ for all positive integers $n \geq 3$. ■

7.4 Prove that for all positive integers $n \geq 4$, $2^n < n!$.

Proof:

Basis Step: Let $n = 4$. Then $2^4 \leq 4! \Leftrightarrow 16 \leq 24$. Since $P(4)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that for all positive integers $n \geq 4$, $2^n < n!$.

Inductive Step: [We must show that $2^{n+1} < (n+1)!$.]

By substituting $n+1$ for n on the right side of the inequality, we obtain $(n+1)! = (n+1)n!$. By our inductive hypothesis, we assume that $2^n(n+1) < (n+1)n! = (n+1)!$ Now we work on the left side. Obviously, $2 < n+1$ for all $n \geq 4$, so we obtain the inequality $2(2^n) < (n+1)2^n \Leftrightarrow 2^{n+1} = 2(2^n) < (n+1)2^n$. By combining the two inequalities, we obtain $2^{n+1} = 2(2^n) < (n+1)2^n < (n+1)n! = (n+1)!$. This proves that $2^{n+1} < (n+1)!$ as desired, making the assertion $P(n+1)$ true. Thus, by the principle of mathematical induction, we have proven that $2^n < n!$ for all positive integers $n \geq 4$. ■

7.5 Prove that for all $x \in \mathbb{R}$ such that $x > -1$, and for all $n \in \mathbb{N}$, $(1+x)^n \geq 1+nx$. (Bernoulli's inequality)

Proof:

Basis Step: Let $n = 0$. Then $(1+x)^0 \geq 1 + 0x \Leftrightarrow 1 \geq 1$. Since $P(0)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that for all real numbers $x > -1$ and for all natural numbers n, $(1+x)^n \geq 1+nx$.

Inductive Step: [We must show that $(1+x)^{n+1} \geq 1 + (n+1)x$.]

By substituting $n+1$ for n on the left side of the inequality, we obtain $(1+x)^{n+1} = (1+x)(1+x)^n$. By our inductive hypothesis, we assume that $(1+x)^{n+1} = (1+x)(1+x)^n \geq (1+x)(1+nx) = 1+nx+x+nx^2$. It is obvious that $1+nx+x+nx^2 \geq 1+nx+x$ for all $n \in \mathbb{N}$, and $1+nx+x = 1+(n+1)x$. Hence, we obtain the inequality $(1+x)^{n+1} = (1+x)(1+x)^n \geq (1+x)(1+nx) = 1+nx+x+nx^2 \geq 1+nx+x = 1+(n+1)x$. This proves that $(1+x)^{n+1} \geq 1+(n+1)x$ as desired, making the assertion $P(n+1)$ true. Thus, by the principle of mathematical induction, we have proven that $(1+x)^n \geq 1+nx$ for all natural integers n and all $x > -1$. ■

8 Other proofs

8.1 Prove that a set with n elements has 2^n subsets.

Proof:

Basis Step: Let S be the empty set. Then S has one subset, $\{\emptyset\}$. Likewise, $2^0 = 1$. Since $P(0)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that a set with n elements has 2^n subsets.

Inductive Step: [We need to prove that a set with $n+1$ elements has 2^{n+1} subsets.] Let T be a set with $n+1$ elements. You can construct the subsets of T in two steps. The first step is that you list the subsets that can be created from the first n elements. By inductive hypothesis, there are 2^n of these subsets. The second step is that you create the remaining subsets by adding the $n+1$ element to the first group of subsets. This would create an additional 2^n subsets. So T has $2^n + 2^n$ subsets, and $2^n + 2^n = 2(2^n) = 2^{n+1}$, making the assertion $P(n+1)$ true. Thus, by the principle of mathematical induction, we have proven that a set with n elements has 2^n subsets. ■

8.2 Prove that $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + x^{n-3} + ... + x^2 + x + 1)$ for all $n \in \mathbb{N}$.

Proof:

Basis Step: Let $n = 0$. Then $x^0 - 1 = 1 - 1 = 0$. Likewise, $(x^0 - 1)(1) = (1-1)(1) = 0(1) = 0$. Since $P(0)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + x^{n-3} + ... + x^2 + x + 1)$ for all $n \in \mathbb{N}$.

Inductive Step: [We must prove that $x^{n+1} - 1 = (x-1)(x^n + x^{n-1} + x^{n-2} + x^{n-3} + ... + x^2 + x + 1)$.]

By substituting $n+1$, we obtain $x^{n+1} - 1 = x(x^n) - 1 = (x-1+1)(x^n) - 1 = (x-1)x^n + x^n - 1$. By our inductive hypothesis, we obtain $(x-1)x^n + (x-1)(x^{n-1}+x^{n-2}+x^{n-3}+...+x^2+x+1) = (x-1)(x^n+x^{n-1}+$

$x^{n-2}+x^{n-3}+...+x^2+x+1)$. So we have shown that $x^{n+1}-1=(x-1)(x^n+x^{n-1}+x^{n-2}+x^{n-3}+...+x^2+x+1)$ as desired, making the assertion $P(n+1)$ true. Thus, by the principle of mathematical induction, we have proven that $x^n-1=(x-1)(x^{n-1}+x^{n-2}+x^{n-3}+...+x^2+x+1)$ for all $n \in \mathbb{N}$. ■

8.3 Prove that $\prod_{i=2}^{n}(1-\frac{1}{i^2})=\frac{n+1}{2n}$ for all integers $n \geq 2$.

Proof:

Basis Step: Let $n=2$. Then $\prod_{i=2}^{2}(1-\frac{1}{i^2})=1-\frac{1}{2^2}=1-\frac{1}{4}=\frac{3}{4}$. Likewise, $\frac{2+1}{2(2)}=\frac{3}{4}$. Since $P(2)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that $\prod_{i=2}^{n}(1-\frac{1}{i^2})=\frac{n+1}{2n}$ all integers $n \geq 2$.

Inductive Step: [We must prove that $\prod_{i=2}^{n+1}(1-\frac{1}{i^2})=\frac{n+2}{2(n+1)}$.]

By substituting $n+1$, we obtain $\prod_{i=2}^{n+1}(1-\frac{1}{i^2})=\prod_{i=2}^{n}(1-\frac{1}{i^2})(1-\frac{1}{(n+1)^2})$. By our inductive hypothesis, we obtain $(\frac{n+1}{2n})(1-\frac{1}{(n+1)^2})=\frac{n+1}{2n}-\frac{n+1}{2n(n+1)^2}=\frac{n+1}{2n}-\frac{1}{2n(n+1)}=\frac{n+1}{2n}\cdot\frac{n+1}{n+1}-\frac{1}{2n(n+1)}=\frac{(n+1)^2}{2n(n+1)}-\frac{1}{2n(n+1)}=\frac{n^2+2n+1-1}{2n(n+1)}=\frac{n^2+2n}{2n(n+1)}=\frac{n(n+2)}{2n(n+1)}=\frac{n+2}{2(n+1)}$. So we have shown that $\prod_{i=2}^{n+1}(1-\frac{1}{i^2})=\frac{n+2}{2(n+1)}$ as desired, making the assertion $P(n+1)$ true. Thus, by the principle of mathematical induction, we have proven that $\prod_{i=2}^{n}(1-\frac{1}{i^2})=\frac{n+1}{2n}$ for all integers $n \geq 2$. ■

8.4 Prove that $a^m a^n = a^{m+n}$ for all $n \in \mathbb{N}$.

Proof:

Basis Step: Let $n=0$. Then $a^m a^0 = a^m \cdot 1 = a^m$. Likewise, $a^{m+0}=a^m$. Since $P(0)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that $a^m a^n = a^{m+n}$ for all $n \in \mathbb{N}$.

Inductive Step: [We must prove that $a^m a^{n+1} = a^{m+(n+1)}$.]

By substituting $n+1$, we obtain $a^m a^{n+1} = a^m a(a^n) = aa^m a^n$. By our inductive hypothesis, we obtain $aa^{m+n}=a^{m+n+1}=a^{m+(n+1)}$. So we have shown that $a^m a^{n+1}=a^{m+(n+1)}$ as desired, making the assertion $P(n+1)$ true. Thus, by the principle of mathematical induction, we have proven that $a^m a^n = a^{m+n}$ for all $n \in \mathbb{N}$. ■

8.5 Prove that $(a^m)^n = a^{mn}$ for all $n \in \mathbb{N}$.

Proof:

Basis Step: Let $n=0$. Then $(a^m)^0=1$. Likewise, $a^{m(0)}=a^0=1$. Since $P(0)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that $(a^m)^n=a^{mn}$ for all $n \in \mathbb{N}$.

Inductive Step: [We must prove that $(a^m)^{n+1}=a^{m(n+1)}$.]

By substituting $n+1$, we obtain $(a^m)^{n+1}=a^m(a^m)^n$. By our inductive hypothesis, we obtain $a^m a^{mn} = a^{mn+m}=a^{m(n+1)}$. So we have shown that $(a^m)^{n+1}=a^{m(n+1)}$ as desired, making the assertion $P(n+1)$ true. By the principle of mathematical induction, we have proven that $(a^m)^n=a^{mn}$ for all $n \in \mathbb{N}$. ■

8.6 Prove that $(ab)^n = a^n b^n$ for all $n \in \mathbb{N}$.

Proof:

Basis Step: Let $n = 0$. Then $(ab)^0 = 1$. Likewise, $a^0 b^0 = 1 \cdot 1 = 1$. Since $P(0)$ is true, we may assume that $P(n)$ is true. Hence, we may form the inductive hypothesis that $(ab)^n = a^n b^n$ for all $n \in \mathbb{N}$.

Inductive Step: [We must prove that $(ab)^{n+1} = a^{n+1} b^{n+1}$.]

By substituting $n + 1$, we obtain $(ab)^{n+1} = ab(ab)^n$. By our inductive hypothesis, we obtain $aba^n b^n = aa^n b^n b = a^{n+1} b^{n+1}$. So we have shown that $(ab)^{n+1} = a^{n+1} b^{n+1}$ as desired, making the assertion $P(n + 1)$ true. By the principle of mathematical induction, we have proven that $(ab)^n = a^n b^n$ for all $n \in \mathbb{N}$. ■