Proofs Involving Relations and Partitions

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Theorem 1: **Let X be a nonempty set and let $\sim$ be an equivalence relation on X. For all $x, y \in X$, $E_x = E_y$ if and only if $x \sim y$.**

PROOF : Since we have a bi-conditional statement, we must prove two implications.

Claim #1: If $E_x = E_y$, then $x \sim y$.

Proof of Claim #1: Suppose that $E_x = E_y$. Let $z \in E_x$. Since $E_x = E_y$, then $z \in E_y$. So we have that $x \sim z$ and $y \sim z$. By symmetry, $x \sim z$ and $z \sim y$. By transitivity, we have that $x \sim y$. $\square$

Claim #2: If $x \sim y$, then $E_x = E_y$.

Proof of Claim #2:

$\Rightarrow$[We need to prove that $E_x \subseteq E_y$.] Suppose that $x \sim y$. This implies that for any $z \in E_x$ we have $x \sim z$. By symmetry, $z \sim x$. Since $z \sim x$ and $x \sim y$ and we have that $z \sim y$ by transitivity. So $z \in E_y$. Hence, $E_x \subseteq E_y$.

$\Leftarrow$[We need to prove that $E_y \subseteq E_x$.] Suppose that $y \sim x$. This implies that for any $z \in E_y$ we have $y \sim z$. By symmetry, $z \sim y$. Since $z \sim y$ and $y \sim x$, we have that $z \sim x$ by transitivity. So $z \in E_x$. Hence, $E_y \subseteq E_x$. $\square$

Therefore, by Claim #1 and Claim #2, we have proven that $E_x = E_y$ if and only if $x \sim y$. $\blacksquare$

Theorem 2: **Let X be a nonempty set and let $\sim$ be an equivalence relation on X, where $E_x = \{z \in X : x \sim z\}$. For two arbitrary elements x and y in X, if $E_x \cap E_y \neq \emptyset$, then $E_x = E_y$.**

Proof:

$\Rightarrow$[We need to prove that $E_x \subseteq E_y$.] Let $z \in E_x$. Then $x \sim z$. Since we are assuming that $E_x \cap E_y \neq \emptyset$, we can say that there exists w such that $w \in E_x \cap E_y$. By definition of intersection, we know that $w \in E_x$. This implies that $x \sim w$. By the definition of intersection, we also know that $w \in E_y$. This implies that $y \sim w$. By symmetry, we can say that $w \sim x$. Since $y \sim w$, $w \sim x$, and $x \sim z$, then by transitivity, we can say that $y \sim z$. Hence we have shown that $E_x \subseteq E_y$.

$\Leftarrow$[We need to prove that $E_y \subseteq E_x$.] Let $z \in E_y$. Then $y \sim z$. Since we are assuming that $E_x \cap E_y \neq \emptyset$, we can say that there exists w such that $w \in E_x \cap E_y$. By definition of intersection, we know that $w \in E_x$. This implies that $x \sim w$. By the definition of intersection, we also know that $w \in E_y$. This implies that $y \sim w$. By symmetry, we can say that $w \sim y$. Since $x \sim w$, $w \sim y$, and $y \sim z$, then by transitivity, we can say that $x \sim z$. Hence we have shown that $E_y \subseteq E_x$.

Therefore, $E_x = E_y$.$\blacksquare$

Theorem 3: **If X is the nonempty universal set and if E is an equivalence relation on X, then the family $\{E_x | x \in X\}$ is a partition of X.**

Proof:

E_x has at least one element, x, for all $x \in X$. So E_x is nonempty, satisfying the first property of partitions. Next, every $x \in X$ belongs to E_x, so by the inclusion of union, we can say that x belongs to $\bigcup_{x \in X} E_x$. So $\bigcup_{x \in X} E_x = X$, which satisfies the second property. Now, by *Theorem 2*, we have already shown that if $E_x \cap E_y \neq \emptyset$, then $E_x = E_y$. Hence, the third property is satified. Therefore, $\{E_x | x \in X\}$ is a partition of X. ■