Proofs Involving Sets

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1 Basic Set Operations

Prove that $(A \cap B)^{\mathsf{C}} = A^{\mathsf{C}} \cup B^{\mathsf{C}}$.

Proof: Let $x \in (A \cap B)^{\mathsf{C}}$. By definition of complement, we can say that $x \notin A \cap B$. This is logically equivalent to $\neg(p \wedge q)$. DeMorgan's Law states that $\neg(p \wedge q) \equiv \neg p \vee \neg q$, so the statement $x \notin A \cap B$ is logically equivalent to $x \notin A$ or $x \notin B$. By definition of complement, $x \in A^{\mathsf{C}}$ or $x \in B^{\mathsf{C}}$. By definition of union, we have that $x \in A^{\mathsf{C}} \cup B^{\mathsf{C}}$. Thus we have proven that $(A \cap B)^{\mathsf{C}} = A^{\mathsf{C}} \cup B^{\mathsf{C}}$. ■

Prove that $A \subseteq B$ if and only if $B^{\mathsf{C}} \subseteq A^{\mathsf{C}}$.

Proof: The statement $A \subseteq B$ implies that if $x \in A$, then $x \in B$. The statement $B^{\mathsf{C}} \subseteq A^{\mathsf{C}}$ implies that if $x \notin B$, then $x \notin A$. Let p represent the statement $x \in A$ and let q represent the statement $x \in B$. Hence, the statement $A \subseteq B$ could be represented as $p \Rightarrow q$ and the statement $B^{\mathsf{C}} \subseteq A^{\mathsf{C}}$ could be represented as $\neg q \Rightarrow \neg p$. Clearly, the second statement is the contrapositive of the first, and hence, $A \subseteq B$ is logically equivalent to $B^{\mathsf{C}} \subseteq A^{\mathsf{C}}$. ■

Prove that $(A^{\mathsf{C}})^{\mathsf{C}} = A$.

Proof:

$\Rightarrow$[We need to prove that $(A^{\mathsf{C}})^{\mathsf{C}} \subseteq A$.] Let $x \in (A^{\mathsf{C}})^{\mathsf{C}}$. By definition of complement, we can say that $x \notin A^{\mathsf{C}}$. By definition of complement, we can say that $x \in A$. Thus we have shown that $(A^{\mathsf{C}})^{\mathsf{C}} \subseteq A$.
$\Leftarrow$[We need to prove that $A \subseteq (A^{\mathsf{C}})^{\mathsf{C}}$.] Let $x \in A$. By definition of complement, we can say that $x \notin A^{\mathsf{C}}$. By definition of complement, we can say that $x \in (A^{\mathsf{C}})^{\mathsf{C}}$. Thus we have shown that $A \subseteq (A^{\mathsf{C}})^{\mathsf{C}}$.
Therefore, $(A^{\mathsf{C}})^{\mathsf{C}} = A$. ■

Prove that $(A \cup B) \cap B^{\mathsf{C}} \subseteq A$.

Proof: Let $x \in (A \cup B) \cap B^{\mathsf{C}}$. By definition of intersection, we can say that $x \in A \cup B$ and $x \in B^{\mathsf{C}}$. By the definition of union, we can say that $x \in A$ or $x \in B$ and $x \in B^{\mathsf{C}}$. Let p represent the statement "$x \in A$" and q represent the statement "$x \in B$". Logically speaking, we have the statements $p \vee q$ and $\neg q$. By elimination, only p is true. Hence, $x \in A$, and so we have shown that $(A \cup B) \cap B^{\mathsf{C}} \subseteq A$. ■

Prove that $A \cap (B \cup C) = (A \cap B) \cup (B \cap C)$.

Proof: By the definition of intersection, we know that $x \in A$ and $x \in B \cup C$. By definition of union, we can say that $x \in A$ and $x \in B$ or $x \in C$. By the distributive law of logic, we can say that $x \in A$ and $x \in B$ or $x \in A$ and $x \in C$. By the definition of intersection, we can say that $x \in A \cap B$ or $x \in B \cap C$. By definition of union, we can say that $x \in (A \cap B) \cup (B \cap C)$. Hence, we have that $A \cap (B \cup C) = (A \cap B) \cup (B \cap C)$ as desired. ■

Let A and B be arbitrary sets. Prove that if $A \subseteq B$, then $A \cup B = B$.

Proof:

$\Rightarrow$[We need to prove that $A \cup B \subseteq B$.] Let x be an element of $A \cup B$.

Case 1: Let $x \in A$. Since $A \subseteq B$, we can say that $x \in B$.

Case 2: Let $x \in B$. Then $x \in B$.

In both cases, we have that $x \in B$. Thus we have proven that $A \cup B \subseteq B$.

$\Leftarrow$[We need to prove that $B \subseteq A \cup B$.] Let x be an element of B. By definition of union, we can say that $x \in A \cup B$. Thus we have shown that $B \subseteq A \cup B$.
Therefore, $A \cup B = B$. ■

2 Indexed Families of Sets

Prove that $A \cap (\bigcup_{i \in I} B_i) = \bigcup_{i \in I}(A \cap B_i)$.

Proof:

$\Rightarrow$[We need to prove that $A \cap (\bigcup_{i \in I} B_i) \subseteq \bigcup_{i \in I}(A \cap B_i)$.] Let x be an element of $A \cap (\bigcup_{i \in I} B_i)$. By definition of intersection, we can say that $x \in A$ and $x \in \bigcup_{i \in I} B_i$. We can conclude that $x \in B_i$ for some $i \in I$. Thus, we have $x \in \bigcup_{i \in I}(A \cap B_i)$. So we have shown that $A \cap (\bigcup_{i \in I} B_i) \subseteq \bigcup_{i \in I}(A \cap B_i)$.
$\Leftarrow$[We need to prove that $(\bigcup_{i \in I} A \cap B_i) \subseteq A \cap (\bigcup_{i \in I} B_i)$.] Let x be an element of $\bigcup_{i \in I}(A \cap B_i)$. We can then say that $x \in A \cap B_i$ for some $i \in I$. By definition of intersection, $x \in A$ and $x \in B_i$. Since $x \in B_i$, we can say that $x \in \bigcup_{i \in I} B_i$. So we have $x \in A$ and $x \in \bigcup_{i \in I} B_i$. Thus, $x \in A \cap (\bigcup_{i \in I} B_i)$. So we have shown that $\bigcup_{i \in I}(A \cap B_i) \subseteq A \cap (\bigcup_{i \in I} B_i)$.
Therefore, $A \cap (\bigcup_{i \in I} B_i) = \bigcup_{i \in I}(A \cap B_i)$. ■

Prove that $(\bigcap_{i \in I} A_i)^{\complement} = \bigcup_{i \in I}(A_i{}^{\complement})$.

Proof:

$\Rightarrow$[We need to prove that $(\bigcap_{i \in I} A_i)^{\complement} \subseteq \bigcup_{i \in I}(A_i{}^{\complement})$.] Let x be an element of $(\bigcap_{i \in I} A_i)^{\complement}$. By definition of complement, we can say that $x \notin \bigcap_{i \in I} A_i$. This implies that $x \notin A_i$ for any $i \in I$. Hence, we can say that $x \in A_i{}^{\complement}$. By definition of union, $x \in \bigcup_{i \in I}(A_i{}^{\complement})$. So we have shown that $(\bigcap_{i \in I} A_i)^{\complement} \subseteq \bigcup_{i \in I}(A_i{}^{\complement})$.
$\Leftarrow$[We need to prove that $\bigcup_{i \in I}(A_i{}^{\complement}) \subseteq (\bigcap_{i \in I} A_i)^{\complement}$.] Let x be an element of $\bigcup_{i \in I}(A_i{}^{\complement})$. We can say that $x \in A_i{}^{\complement}$ for some $i \in I$. By definition of complement, $x \notin A_i$. By definition of intersection, $x \notin \bigcap_{i \in I} A_i$. By definition of complement, $x \in (\bigcap_{i \in I} A_i)^{\complement}$. So we have shown that $\bigcup_{i \in I}(A_i{}^{\complement}) \subseteq (\bigcap_{i \in I} A_i)^{\complement}$.
Therefore, $(\bigcap_{i \in I} A_i)^{\complement} = \bigcup_{i \in I}(A_i{}^{\complement})$. ■

Let $\{A_i | i \in I\}$ and $\{B_i | i \in I\}$ be indexed families of sets such that $I \neq \emptyset$. Prove that $\bigcup_{i \in I}(A_i \setminus B_i) \subseteq (\bigcup_{i \in I} A_i) \setminus (\bigcap_{i \in I} B_i)$.

Proof: Suppose that $x \in \bigcup_{i \in I}(A_i \setminus B_i)$. Then for some $i \in I$, $x \in A_i \setminus B_i$. This means that $x \in A_i$ and $x \notin B_i$. Since $x \in A_i$, then $x \in \bigcup_{i \in I} A_i$, and since $x \notin B_i$, then $x \notin \bigcap_{i \in I} B_i$. Thus, we can say that $x \in (\bigcup_{i \in I} A_i) \setminus (\bigcap_{i \in I} B_i)$. ■

3 Power Sets

Prove that $\wp(A \cap B) = \wp(A) \cap \wp(B)$.

Proof:

⇒[We need to prove that $\wp(A \cap B) \subseteq \wp(A) \cap \wp(B)$.] Let C be an element of $\wp(A \cap B)$. By definition of power set, we can say that $C \subseteq A \cap B$. By definition of intersection, $C \subseteq A$ and $C \subseteq B$. By definition of power set, $C \in \wp(A)$ and $C \in \wp(B)$. By definition of intersection, $C \in \wp(A) \cap \wp(B)$. Thus we have shown that $\wp(A \cap B) \subseteq \wp(A) \cap \wp(B)$.
⇐[We need to prove that $\wp(A) \cap \wp(B) \subseteq \wp(A \cap B)$.] Let C be an element of $\wp(A) \cap \wp(B)$. By definition of intersection, we can say that $C \in \wp(A)$ and $C \in \wp(B)$. By definition of power set, $C \subseteq A$ and $C \subseteq B$. By definition of intersection, $C \subseteq A \cap B$. By definition of power set, $C \in \wp(A \cap B)$. Thus we have shown that $\wp(A) \cap \wp(B) \subseteq \wp(A \cap B)$.
Therefore, $\wp(A \cap B) = \wp(A) \cap \wp(B)$. ■

Prove that $\wp(A) \cup \wp(B) \subseteq \wp(A \cup B)$.

Proof Let C be an element of $\wp(A) \cup \wp(B)$. By definition of union, we can say that $C \in \wp(A)$ or $C \in \wp(B)$. By definition of power set, $C \subseteq A$ or $C \subseteq B$.

Case #1: Let $C \subseteq A$. By generalization, $C \subseteq A \cup B$.

Case #2: Let $C \subseteq B$. By generalization, $C \subseteq A \cup B$.

In either case, we have that $C \subseteq A \cup B$. By definition of power set, $C \in \wp(A \cup B)$. Thus we have shown that $\wp(A) \cup \wp(B) \subseteq \wp(A \cup B)$. ■

Prove that $\bigcup_{i \in I} \wp(A_i) \subseteq \wp(\bigcup_{i \in I} A_i)$.

Proof Let B be an element of $\bigcup_{i \in I} \wp(A_i)$. This implies that $B \in \wp(A_i)$ for some $i \in I$. By definition of power set, $B \subseteq A_i$ for some $i \in I$. By generalization, $B \subseteq \bigcup_{i \in I} A_i$. By definition of power set, $B \in \wp(\bigcup_{i \in I} A_i)$. Thus we have shown that $\bigcup_{i \in I} \wp(A_i) \subseteq \wp(\bigcup_{i \in I} A_i)$. ■

4 Cartesian Products of Sets

Theorem 1: For all sets A, B, and C, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

PROOF:

⇒[We need to prove that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.] Let z be an element of $A \times (B \cup C)$. By definition of Cartesian product, $z = (x, y)$ for some x in A and y in $B \cup C$. By definition of union, $y \in B$ or $y \in C$.

Case 1: Let $y \in B$. Since x is in A, then (x, y) is in $A \times B$ by definition of Cartesian product. By generalization, $(x, y) \in A \times B$ or $(x, y) \in A \times C$. By definition of union, $(x, y) \in (A \times B) \cup (A \times C)$.

Case 2: Let $y \in C$. Since x is in A, then (x, y) is in $A \times C$ by definition of Cartesian product. By generalization, $(x, y) \in A \times B$ or $(x, y) \in A \times C$. By definition of union, $(x, y) \in (A \times B) \cup (A \times C)$.

In both cases, we see that $(x, y) \in (A \times B) \cup (A \times C)$. Since $z = (x, y)$, we can say that $z \in (A \times B) \cup (A \times C)$. Thus, we have shown that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$. $\square$

$\Leftarrow$[We need to prove that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.] Let z be an element of $(A \times B) \cup (A \times C)$. By definition of union, $z \in (A \times B)$ or $z \in (A \times C)$.

Case 1: Let $z \in (A \times B)$. By definition of Cartesian product, $z = (x, y)$ for some x in A and y in B. By generalization, $y \in B$ or $y \in C$. By definition of union, $y \in B \cup C$. So we have that $(x, y) \in A \times (B \cup C)$.

Case 2: Let $z \in (A \times C)$. By definition of Cartesian product, $z = (x, y)$ for some x in A and y in C. By generalization, $y \in B$ or $y \in C$. By definition of union, $y \in B \cup C$. So we have that $(x, y) \in A \times (B \cup C)$.

In both cases, we have that $(x, y) \in A \times (B \cup C)$. Since $z = (x, y)$, we have that $z \in A \times (B \cup C)$. Thus we have proven that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$. $\square$

Therefore, $A \times (B \cup C) = (A \times B) \cup (A \times C)$. $\blacksquare$

Theorem 2: For all non-empty sets A and B, $A \times B = B \times A$ if and only if $A = B$.

PROOF : Since we have a bi-conditional statement, we must prove two claims.

Claim #1: If $A \times B = B \times A$, then $A = B$.

Proof of Claim #1:

$\Rightarrow$[We need to prove that $A \subseteq B$.] Let $a \in A$ and $b \in B$. By definition of Cartesian product, we know that $(a, b) \in A \times B$. Assuming that $A \times B = B \times A$, we can say that $(a, b) \in B \times A$. This implies that $a \in B$. Since $a \in A$ implies $b \in B$, we have shown that $A \subseteq B$.

$\Leftarrow$[We need to prove that $B \subseteq A$.] Let $b \in B$ and $a \in A$. By definition of Cartesian product, we know that $(a, b) \in A \times B$. Assuming that $A \times B = B \times A$, we can say that $(a, b) \in B \times A$. This implies that $b \in A$. Since $b \in B$ implies $b \in A$, we have shown that $B \subseteq A$.

So we have that $A = B$. Thus, we have proven that if $A \times B = B \times A$, then $A = B$. $\square$

Claim #2: If $A = B$, then $A \times B = B \times A$.

Proof of Claim #2:

Given that $A = B$, we can say that $A \times B = A \times A = B \times A$. Thus we have proven that if $A = B$, then $A \times B = B \times A$. $\square$

Therefore, by Claim #1 and Claim #2, we can say that $A \times B = B \times A$ if and only if $A = B$. $\blacksquare$

Theorem 3: For any set A, $A \times \emptyset = \emptyset$.

PROOF:

Suppose for contradiction that $A \times \emptyset \neq \emptyset$. Then there exists an element $(x, y) \in A \times \emptyset$. By definition of Cartesian product, this implies that $x \in A$ and $y \in \emptyset$. However, this contradicts the fact that $\emptyset$ is the empty set. Thus $A \times \emptyset = \emptyset$. $\blacksquare$