Proofs Involving Functions

July 23, 2009

***Theorem 1*: Define f by $f : X \to Y$ such that $A \subseteq X$ and $B \subseteq X$. Then $f(A \cup B) = f(A) \cup f(B)$.**

Proof:

$\Rightarrow$[We need to prove that $f(A \cup B) \subseteq f(A) \cup f(B)$.] Let y be an element of $f(A \cup B)$. Then there exists an x in $A \cup B$ such that $f(x) = y$. By definition of union, $x \in A$ or $x \in B$. Thus implies that $f(x) \in f(A)$ or $f(x) \in f(B)$. Since $f(x) = y$, we have that $y \in f(A) \cup f(B)$. Thus $f(A \cup B) \subseteq f(A) \cup f(B)$.
$\Leftarrow$[We need to prove that $f(A) \cup f(B) \subseteq f(A \cup B)$.] Let y be an element of $f(A) \cup f(B)$. Then $y \in f(A)$ or $y \in f(B)$.

Case 1: Let $y \in f(A)$. Then there exists an x in A such that $f(x) = y$. By inclusion of union, $x \in A \cup B$. Thus, $f(x) \in f(A \cup B)$. Since $f(x) = y$, we have that $y \in f(A \cup B)$.

Case 2: Let $y \in f(B)$. Then there exists an x in B such that $f(x) = y$. By inclusion of union, $x \in A \cup B$. Thus, $f(x) \in f(A \cup B)$. Since $f(x) = y$, we have that $y \in f(A \cup B)$.

In both cases, we have that $y \in f(A \cup B)$. Thus we have proven that $f(A) \cup f(B) \subseteq f(A \cup B)$.

Therefore, $f(A \cup B) = f(A) \cup f(B)$. $\blacksquare$

***Theorem 2*: Define f by $f : X \to Y$ such that $A \subseteq X$ and $B \subseteq X$. Then $f(A \cap B) \subseteq f(A) \cup f(B)$.**

Proof:

Let y be an element of $f(A \cap B)$. Then there exists an x in $A \cap B$ such that $f(x) = y$. By definition of intersection, $x \in A$ and $x \in B$. Thus implies that $f(x) \in f(A)$ and $f(x) \in f(B)$. Since $f(x) = y$, we have that $y \in f(A) \cap f(B)$. Thus $f(A \cap B) \subseteq f(A) \cap f(B)$. $\blacksquare$

***Theorem 3*: Define f by $f : X \to Y$ such that $B \subseteq Y$. Then $f^{-1}(B^{\complement}) = (f^{-1}(B))^{\complement}$.**

Proof:

$\Rightarrow$[We need to prove that $f^{-1}(B^{\complement}) \subseteq (f^{-1}(B))^{\complement}$.] Let x be an element of $f^{-1}(B^{\complement})$. Then for some $y \in B^{\complement}$, $f^{-1}(y) = x$. By definition of complement, $y \notin B$. This implies that $f^{-1}(y) \notin f^{-1}(B)$. By definition of complement, $f^{-1}(y) \in (f^{-1}(B))^{\complement}$. Recalling that $f^{-1}(y) = x$, we have that $x \in (f^{-1}(B))^{\complement}$. Thus, $f^{-1}(B^{\complement}) \subseteq (f^{-1}(B))^{\complement}$.
$\Leftarrow$[We need to prove that $(f^{-1}(B))^{\complement} \subseteq f^{-1}(B^{\complement})$.] Let x be an element of $(f^{-1}(B))^{\complement}$. By definition of complement, $x \notin f^{-1}(B)$. By definition of inverse image, there exists a y such that $x = f^{-1}(y)$, implying that $f^{-1}(y) \notin f^{-1}(B)$. This implies that $y \notin B$. By definition of complement, we have that $y \in B^{\complement}$, implying that $f^{-1}(y) \in f^{-1}(B^{\complement})$. Recalling that $x = f^{-1}(y)$, we have that $x \in f^{-1}(B^{\complement})$. Thus, $(f^{-1}(B))^{\complement} \subseteq f^{-1}(B^{\complement})$.
Therefore, $f^{-1}(B^{\complement}) = (f^{-1}(B))^{\complement}$. $\blacksquare$

***Theorem 4*: Define f by $f : X \to Y$ and define g by $g : Y \to Z$. If f and g are injective, then $g \circ f$ is injective.**

Proof: Let $a, b \in X$ where $a \neq b$. By definition of injective, $f(a) \neq f(b)$. Since g is one to one, $g(f(a)) \neq g(f(b))$. Therefore, $g \circ f$ is injective. ■

***Theorem 5*: Define f by $f : X \to Y$ and define g by $g : Y \to Z$. Prove that $g \circ f$ maps X onto Z.**

Proof: Let $z \in Z$. Since g is onto, there exists $y \in Y$ such that $g(y) = z$. Since f is onto, there exists $x \in X$ such that $f(x) = y$. So $g \circ f = g(f(x)) = g(y) = z$. Thus, $g \circ f$ maps X onto Z. ■

***Theorem 6*: Let f be a bijection from X onto Y. Let g be a bijection from Y onto Z. Prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.**

Proof: Since f and g are bijective, we have already established that $g \circ f$ maps X onto Z. This implies that f, g, and $g \circ f$ have inverse functions. Thus, $(g \circ f)^{-1}$ maps Z onto X. Let $z \in Z$ and let $(g \circ f)^{-1}(z) = x$. Then by definition of inverse, we can say that $(g \circ f)(x) = z$. By definition of composition, $g(f(x) = z$. By the definition of inverse of g, $g^{-1}(z) = f(x)$. By definition of the inverse of f, $f^{-1}(g^{-1}(z)) = x$. By definition of composition, we can say that $(f^{-1} \circ g^{-1})(z) = x$. So, we have shown that $(g \circ f)^{-1}(z) = (f^{-1} \circ g^{-1})(z)$. Thus, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. ■

***Theorem 7*: Define f by $f : A \to B$. Then $f(\bigcup\limits_{i \in I} V_i) = \bigcup\limits_{i \in I} f(V_i)$.**

Proof:

$\Rightarrow$[We need to prove that $f(\bigcup\limits_{i \in I} V_i) \subseteq \bigcup\limits_{i \in I} f(V_i)$.] Let b be an element of $f(\bigcup\limits_{i \in I} V_i)$. Then there exists an element v in $\bigcup\limits_{i \in I} V_i$ such that $f(v) = b$. By definition of union, $v \in V_i$ for some $i \in I$. This implies that $b \in f(V_i)$. By inclusion of union, we can say that $b \in \bigcup\limits_{i \in I} f(V_i)$. Thus $f(\bigcup\limits_{i \in I} V_i) \subseteq \bigcup\limits_{i \in I} f(V_i)$.
$\Leftarrow$[We need to prove that $\bigcup\limits_{i \in I} f(V_i) \subseteq f(\bigcup\limits_{i \in I} V_i)$.] Let a be an element of $\bigcup\limits_{i \in I} f(V_i)$. Then $a \in f(V_i)$ for some $i \in I$. Thus, we can say that $a = f(v)$ for some $v \in V_i$. By inclusion of union, $v \in \bigcup\limits_{i \in I} V_i$. Hence, we can say that $a \in f(\bigcup\limits_{i \in I} V_i)$. Thus, $\bigcup\limits_{i \in I} f(V_i) \subseteq f(\bigcup\limits_{i \in I} V_i)$.
Therefore, $f(\bigcup\limits_{i \in I} V_i) = \bigcup\limits_{i \in I} f(V_i)$. ■

***Theorem 8*: Define f by $f : A \to B$. Let $S, T \subseteq B$. If $S \subseteq T$, then $f^{-1}(S) \subseteq f^{-1}(T)$.**

Proof: If x is an element of $f^{-1}(S)$, then $f(x) \in S$. By definition of subset, we can say that $f(x) \in T$. It follows that $x \in f^{-1}(T)$. Thus, $f^{-1}(S) \subseteq f^{-1}(T)$. ■

***Theorem 9*: Define f by $f : X \to Y$ such that $A \subseteq X$. Then $A \subseteq f^{-1}(f(A))$.**

Proof: Let z be an element of A. This implies that $f(z) \in f(A)$. Consequently, $z \in f^{-1}(f(A))$, implying that $A \subseteq f^{-1}(f(A))$. ■

Let $f(x) = x^2 + 1$, where $x \in \mathbb{R}$. Prove that f maps $\mathbb{R}$ onto $[1, \infty)$.

Proof: Since $x \in \mathbb{R}$, $x^2 \geq 0$, so $f(x) \geq 1$. Let $y \in [1, \infty)$. Then $y \geq 1$, so $y - 1 \geq 0$. Thus, $\sqrt{y-1}$ is a real number. Let $x = \sqrt{y-1}$. Then $f(x) = f(\sqrt{y-1}) = (\sqrt{y-1})^2 + 1 = y - 1 + 1 = y$. Thus, for every $y \in [1, \infty)$, there exists $x \in \mathbb{R}$ such that $f(x) = y$. Thus, f maps $\mathbb{R}$ onto $[1, \infty)$. ■

Let $f(x) = 2x$, where $x \in \mathbb{N}$. Explain why f does not map $\mathbb{N}$ onto $\mathbb{N}$.

Let $y = 3$. By definition of natural number, $3 \in \mathbb{N}$. Since $x \in \mathbb{N}$, we can say that $x \in \mathbb{Z}$ since $\mathbb{N} \subseteq \mathbb{Z}$. Since $f(x) = 2x = y$, then y must be an even number. However, 3 is not an even number. So $f(x) \neq 3$, implying that $f : \mathbb{N} \to \mathbb{N}$ is not onto.